An Involutary Matrix of Eigenvectors

David Callan,¹
Department of Statistics, University of Wisconsin-Madison,
1210 W. Dayton Street, Madison, WI 53706-1693, U.S.A.

Helmut Prodinger,
The John Knopfmacher Centre for Applicable Analysis and Number Theory,
School of Mathematics, University of the Witwatersrand,
P. O. Wits, 2050 Johannesburg, South Africa.

A matrix with a full set of linearly independent eigenvectors is diagonalizable: if the
$n$ by $n$ matrix $A$ has eigenvalues $\lambda_j$ with corresponding eigenvectors $u_j \ (1 \leq j \leq n)$, if
$U = (u_1 | u_2 | \ldots | u_n)$ and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $U$ is a diagonalizing matrix for $A$:
$U^{-1}AU = D$. Taking transposes shows that $(U^{-1})^t$ is a diagonalizing matrix for $A^t$. Hence
$U^t$ itself is a diagonalizing matrix for $A^t$ if $U^2$ is the identity matrix, or more generally, due
to the scalability of eigenvectors, if $U^2$ is a scalar matrix.

The purpose of this note is to point out that the right-justified Pascal-triangle matrix
$R = \binom{i-1}{j-n} \sum_{i=1}^{j} (-1)^{j-i} a^{2i-n-1}$ is an example of this phenomenon. Let $a$ denote the golden ratio
$(1 + \sqrt{5})/2$. The eigenvalues of $R^t$ (which of course are the same as the eigenvalues of $R$)
were found in [1]: $\lambda_i = (-1)^{n-i} a^{2i-n-1}, \ 1 \leq i \leq n$. The corresponding eigenvectors $u_i$ of
$R^t$ were also found in [1] (here suitably scaled for our purposes): $u_i = (u_{ij})_{1 \leq j \leq n}$ where
$u_{ij} = (-a)^{n-j} \sum_{k=1}^{j} (-1)^{j-i} k \binom{i-1}{k} \binom{n-j}{k^2} a^{2i-k-1}.

Let $U = (u_{ij})_{1 \leq i, j \leq n}$.

For example, when $n = 5$,

$$R = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 2 & 1 \\
0 & 1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix} \quad \text{and} \quad
U = \begin{pmatrix}
a^4 & -4a^3 & 6a^2 & -4a & 1 \\
-a^3 & 3a^2-a^4 & -3a+3a^3 & 1-3a^2 & a \\
a^2 & -2a+2a^3 & 1-4a^2+a^4 & 2a-2a^3 & a^2 \\
a & 1-3a^2 & 3a-3a^3 & 3a^2-a^4 & a^3 \\
1 & 4a & 6a^2 & 4a^3 & a^4
\end{pmatrix}.$$
Since the rows of $U$ are eigenvectors of $R^i$, $U^i$ is a diagonalizing matrix for $R^i$. By the first paragraph applied to $A = R^i$, $U$ will be a diagonalizing matrix for $R$ if we can show that $(U^i)^2$ (equivalently $U^2$) is a scalar matrix. We now proceed to show that $U^2 = (1 + a^2)^{n-1} I_n$ and in fact this holds for arbitrary $a$. We use the notation $[x^k]p(x)$ to denote the coefficient of $x^k$ in the polynomial $p(x)$. Consider the generating function $U_i(z) = z(z-a)^{n-i}(az+1)^{i-1}$. Using the binomial therem to expand $U_i(z)$, it is immediate that

$$U_i(z) = \sum_{j=1}^{n} u_{ij} z^j.$$

Now the $(i, k)$ entry of $U^2$ is

$$(U^2)_{ik} = \sum_{j=1}^{n} [x^{k-1}](x-a)^{n-j}(ax+1)^{j-1} \cdot [z^{j-1}](z-a)^{n-i}(az+1)^{i-1}$$

$$= [x^{k-1}]\sum_{j=1}^{n} [z^{j-1}](x-a)^{n-j}(ax+1)^{j-1} (z-a)^{n-i}(az+1)^{i-1}$$

$$= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^{n} [z^{j-1}] \left(\frac{ax+1}{x-a}\right)^{j-1} (z-a)^{n-i}(az+1)^{i-1}$$

$$= [x^{k-1}](x-a)^{n-1} \sum_{j=1}^{n} [z^{j-1}] \left(\frac{ax+1}{x-a} - a\right)^{j-1} \left(\frac{ax+1}{x-a} + 1\right)^{i-1}$$

$$= [x^{k-1}](ax+1-ax+a^2)^{n-1}(a^2x+a+x-a)^{i-1}$$

$$= [x^{k-i}](1+a^2)^{n-1} x^{i-1}(1+a^2)^{i-1}$$

$$= [x^{k-i}] (1 + a^2)^{n-1} \delta_{ki},$$

as desired.

References

AMS Classification Numbers: 11B65, 15A36, 15A18