Another Determinant Condensation Formula

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It is shown that $CD = kDC$ implies that $k^r \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - kBC)$ where $r$ is the total multiplicity of nonzero eigenvalues of $C$. This extends similar results known for the hypothesis $CD^T = \pm DC^T$, but its proof turns out to be more complicated. We also examine when our hypothesis is satisfied.

An Advanced Problem in the American Mathematical Monthly [1] prompted several exchanges on the use of continuity in establishing matrix identities. The idea is, you prove the identity for nonsingular matrices and then make an appeal to continuity to cover the singular case. M. J. Pelling gave a celebrated example [2] where this technique fails, for the very good reason that the result isn't true for singular matrices. In [3] Marvin Marcus showed that continuity is not in fact an issue in Pelling's example by giving a purely algebraic proof of essentially the following result (matrix entries in an integral domain).

**THEOREM 1** Suppose $A, B, C, D$ are $n$ by $n$ matrices satisfying

$$CD^T + DC^T = 0$$

Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^r \det(AD^T + BC^T)$$

where $r = \text{rank}(C)$.

The point here is that the two determinants in (2) are necessarily equal when $D$ is nonsingular but not necessarily equal when $D$ is singular (assuming characteristic $\neq 2$, of course). To see this (and prove Theorem 1) note that (1) and (2) are unchanged when $A, B, C, D$ are replaced by $Q^{-1}AQ, Q^{-1}B(Q^T)^{-1}, PCQ, PD(Q^T)^{-1}$ respectively, for any nonsingular matrices $P, Q$. Hence we may assume without loss of generality that $C = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$ [the Smith normal form].

Let $\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ denote the corresponding block structure of $D$. Then (1) implies that $D_{21} = 0$ and $D_{11}^T = -D_{11}$. Simple determinant manipulations (as in the proof of our main result below, or see [4]) now serve to establish (2). Further, if $D$ is nonsingular then $\det D_{11} \neq 0$ and so the size $r$ of $D_{11}$ is even (since $\det D_{11} = \det D_{11} = \det(-D_{11}) = (-1)^r \det D_{11}$) and the determinants in (2) are actually equal, as claimed.

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While this disposes of the continuity question, it does raise another algebraic question: can anything be said under the hypothesis $CD + DC = 0$? It turns out that an analog does indeed hold for this case. However its proof is a good deal more complicated, mainly because we are now restricted to replacing $C$—while leaving the problem invariant—by the Jordan form $PCP^{-1}$, rather than by the simpler Smith form $PCQ$. The proof of our main result, Theorem 2 below, follows some suggestions of Professor E. C. Dade which substantially improved and simplified the author's original proof.

**Theorem 2** Let $A$, $B$, $C$, $D$ be $n$ by $n$ matrices over an integral domain $R$ with 1. Suppose

$$CD = kDC$$

(3)

for some scalar $k$ in $R$. Then

$$k^r \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - kBC)$$

(4)

where $r$ is the total multiplicity of nonzero eigenvalues of $C$ (and $0^0 := 1$).

**Proof** By embedding $R$ in its field of quotients, we may assume that $R$ is in fact a field. Since both hypothesis and conclusion are unchanged under conjugation of $A$, $B$, $C$, $D$ by a suitable nonsingular matrix $P$, we may assume that $C$ is a block diagonal matrix

$$C = \begin{pmatrix} C_{00} & 0 \\ 0 & C_{11} \end{pmatrix}$$

where $C_{00}$ is nilpotent and $C_{11}$ is nonsingular. Let $n_0$, $n_1$ denote the sizes of $C_{00}$, $C_{11}$ respectively. We have $n_0, n_1 \geq 0, n_0 + n_1 = n$, and of course the number of nonzero eigenvalues is $r = n_1$. Say $(D_{00} \ D_{01})$ is the corresponding block structure of $D$.

Let us first dispose of the cases where $k = 0$. Case (i) $r(= n_1) > 0$. The hypothesis implies $D_{10} = 0$ and $D_{11} = 0$, whence $D$ is singular and both sides of (4) vanish. Case (ii) $r(= n_1) = 0$. Here we must show $(0^0 := 1)$

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D$$

(5)

If $C_{00} (= C)$ is zero, (5) is trivial; if not, the hypothesis $CD = 0$ implies $D$ is singular, hence has a left 0-eigenvector $u$. Choose $i \geq 0$ maximal such that $uc^i \neq 0$ (possible since $C$ is nilpotent). Then $uc^i$ is a common left 0-eigenvector of $C$ and $D$, whence the matrix $(C \ D)$ has deficient row rank and $(c \ b)$ is singular. Thus in this case both sides of (4) vanish.

Henceforth we assume $k \neq 0$.

**Lemma 1** $D_{10} = 0$, $D_{01} = 0$. 


Proof The hypothesis implies $C_{00}D_{01} = kD_{01}C_{11}$ and, by iteration, $C_{0i}D_{01} = k^{i}D_{01}C_{11}$. But $C_{00}$ nilpotent implies $C_{0i}^{i} = 0$ for some $i$, whence $D_{01} = 0$. Similarly $D_{10} = 0$.

Due to Lemma 1 we may conjugate further by an $n$ by $n$ matrix of the form $(p_{0} \ 0 \ i)$ where $P_{00}$ is $n_{0}$ by $n_{0}$ and nonsingular, to assume that

$$D_{00} = \begin{pmatrix} D_{00,00} & 0 \\ 0 & D_{00,11} \end{pmatrix}$$

where $D_{00,00}$ is $n_{00}$ by $n_{00}$ and nilpotent, while $D_{00,11}$ is $n_{11}$ by $n_{11}$ and nonsingular $(n_{00}, n_{11} \geq 0, n_{00} + n_{11} = n_{0})$. Let the corresponding block structure of $C_{00}$ be

$$C_{00} = \begin{pmatrix} C_{00,00} & C_{00,01} \\ C_{00,10} & C_{00,11} \end{pmatrix}.$$

As in Lemma 1, we have $C_{00,01} = 0$ and $C_{00,10} = 0$, whence $C_{00,00}$ is nilpotent since $C_{00}$ is. Thus

$$C = \begin{pmatrix} C_{00,00} & 0 & 0 \\ 0 & C_{00,11} & 0 \\ 0 & 0 & C_{11} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_{00,00} & 0 & 0 \\ 0 & D_{00,11} & 0 \\ 0 & 0 & D_{11} \end{pmatrix} \quad (6)$$

We complete the proof of (4) by considering two cases.

CASE (i) $n_{00} > 0$. From the hypothesis, we have $C_{00,00}^{i}D_{00,00}^{j} = k^{i}D_{00,00}^{j}C_{00,00}$ as in Lemma 1, and then by an argument similar to that of the $k = 0$ case (ii) above, there exists a common left 0-eigenvector $u_{00}$ and also a common right 0-eigenvector $v_{00}$ of $C_{00,00}$ and $D_{00,00}$. Thus by (6),

$$u_{0} := (u_{00}, 0) \quad \text{[resp.} \quad v_{0} := \begin{pmatrix} v_{00} \\ 0 \end{pmatrix} \quad \text{]}$$

is a common left [resp. right] 0-eigenvector of $C$ and $D$. But then $(0, u_{0})(z \ A \ b) = 0$, whence $(z \ A \ b)$ is singular and det$(z \ A \ b) = 0$. Similarly $(AD - kBC)v_{0} = 0$ and det$(AD - kBC) = 0$. Thus in this case again both sides of (4) vanish.

CASE (ii) $n_{00} = 0$. This means we have

$$C = \begin{pmatrix} C_{00} & 0 \\ 0 & C_{11} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_{00} & 0 \\ 0 & D_{11} \end{pmatrix}$$
with $C_{00}$ nilpotent and $C_{11}, D_{00}$ both nonsingular. Determinant manipulations now suffice to establish (4). We have

$$
\text{det} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{det} \begin{pmatrix} A_{00} & A_{01} & B_{00} & B_{01} \\ A_{10} & A_{11} & B_{10} & B_{11} \\ C_{00} & 0 & D_{00} & 0 \\ 0 & C_{11} & 0 & D_{11} \end{pmatrix}
$$

\[= \text{det} \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & A_{01} & B_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & A_{11} & B_{10} & B_{11} - A_{11}C_{11}^{-1}D_{11} \\ 0 & 0 & D_{00} & 0 \\ 0 & 0 & C_{11} & 0 \end{pmatrix}
\]

\[= (-1)^{n_1} \text{det} \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & B_{11} - A_{11}C_{11}^{-1}D_{11} \end{pmatrix} \text{det} D_{00} \text{det} C_{11} \]

(7)

On the other hand, from the hypothesis $CD = kDC$ it follows directly that $D_{00}^{-1} \times C_{00}D_{00} = kC_{00}$ and $kC_{11}^{-1}D_{11}C_{11} = D_{11}$, and we have

$$
\text{det}(AD - kBC) = \text{det} \begin{pmatrix} A_{00}D_{00} - kB_{00}C_{00} & A_{01}D_{11} - kB_{01}C_{11} \\ A_{10}D_{00} - kB_{10}C_{00} & A_{11}D_{11} - kB_{11}C_{11} \end{pmatrix}
$$

\[= \text{det} \begin{pmatrix} A_{00}D_{00} - B_{00}D_{00}^{-1}C_{00}D_{00} & kA_{01}C_{11}^{-1}D_{11}C_{11} - kB_{01}C_{11} \\ A_{10}D_{00} - B_{10}D_{00}^{-1}C_{00}D_{00} & kA_{11}C_{11}^{-1}D_{11}C_{11} - kB_{11}C_{11} \end{pmatrix}
\]

\[= (-k)^{n_1} \text{det} \begin{pmatrix} A_{00} - B_{00}D_{00}^{-1}C_{00} & B_{01} - A_{01}C_{11}^{-1}D_{11} \\ A_{10} - B_{10}D_{00}^{-1}C_{00} & B_{11} - A_{11}C_{11}^{-1}D_{11} \end{pmatrix} \text{det} D_{00} \text{det} C_{11}
\]

(8)

Since $r = n_1$, (4) follows immediately from (7) and (8). This concludes the proof of Theorem 2.

Of course, the matrix $C$ does not have a privileged role over that of $D$. In fact, Theorem 2 also holds defining $r$ to be the total multiplicity of zero eigenvalues of $D$. The proof is much like that of Theorem 2 when $k = 0$; and for $k \neq 0$ simply switch the block columns in $(A \ C \ B \ D)$, apply Theorem 2 to the resulting matrix, then switch back.

Now we may ask: do any nontrivial matrices satisfy $CD = kDC$. The equation $AX = XB$ is fully solved over $C$ in [5, p. 421] and the results imply the following Theorem (for $F = C$).

**Theorem 3** Suppose $C \in F^{n \times n}$ (n by n matrices over $F$) and let $\{p_i(x)^{e_i}\}_{i=1}^r$ denote the elementary divisors of $C$. 
Then the dimension of the vector space \( V := \{ D \in F^{n \times n} : CD = kDC \} \) is given by

\[
\dim V = \sum_{i=1}^{n} \sum_{j=1}^{n} \deg(\gcd(p_i(x)^{e_i}, p_j(x)^{e_j})) \quad k \neq 0 \\
= n \sum_{j=1}^{n} \deg(\gcd(x, p_j(x)^{e_j})) \quad k = 0
\]

(Thus nontrivial \( D \) exist iff some pair of eigenvalues of \( C \) have ratio \( k \).)

Since the spirit of the present paper is algebraic generality, we include a proof that Theorem 3 is in fact true for arbitrary fields \( F \).

**Lemma** Suppose \( A, B \) are square matrices (not necessarily the same size) with entries in \( F \). Then \( AX = XB \) has only the trivial solution for \( X \) unless \( A, B \) have a common eigenvalue (in \( F \)).

**Proof** One method is to realize the matrix equation \( AX = XB \) as a homogeneous linear system of equations for the entries in \( X \) with coefficient matrix \( A \times I - I \times B^T \) (Kronecker products) whose eigenvalues are known to be \( \lambda_A - \lambda_B \) where \( \lambda_A \) ranges over the eigenvalues of \( A \) and similarly for \( \lambda_B \). Another method is to use Jordan canonical forms for \( A \) and \( B \) (permissible since \( AX = XB \)

\[ \iff (PA^{-1}(PXPQ) = (PXP)(Q^{-1}BQ)) \]

This reduces the problem to solving \( (\lambda_A I + J_1)X = X(\lambda_B I + J_2) \) with \( J_1, J_2 \) nilpotent, which is quite easy. The details are left to the reader.

**Proof of Theorem 3** We treat the case \( k \neq 0 \) (leaving \( k = 0 \) as an exercise). We may clearly assume by conjugation that \( C \) is in primary rational canonical form \( C = \text{diag}(C_1, C_2, \ldots, C_t) \) where \( C_i \) has minimum polynomial \( p_i(x)^{e_i} \), \( p_i \) irreducible over \( F \).

Partitioning \( D \) into corresponding blocks \( D_{ij} \), the hypothesis \( CD = kDC \) is equivalent to \( C_i D_{ij} = kD_{ij} C_j \) for all \( i, j \). We count the contribution of each of these equations to \( \dim V \). Now \( \deg(\gcd(p_i(x)^{e_i}, p_j(x)^{e_j})) = 0 \) iff \( \gcd(p_i(x), p_j(x)) = 1 \) iff \( C_i \) and \( kC_j \) have no common eigenvalue, and this implies \( D_{ij} = 0 \) by the lemma.

On the other hand \( \deg(\gcd(p_i(x)^{e_i}, p_j(x)^{e_j})) > 0 \) iff \( p_i(x) = k_{\deg p_i} p_j(x) \) (since \( p_i, p_j \) are irreducible), from which it follows that the minimum polynomial of \( kC_j \)

\[ p_i(x)^{e_i}, \text{ and } \deg(\gcd(p_i(x)^{e_i}, p_j(x)^{e_j})) = \deg p_i^{\min(e_i, e_j)} \]

Thus, by conjugating, we may assume [6, p. 203 Ex. 13] that \( C_i = P_i \) and \( kC_j = P_j \) where \( P_m \) denotes the matrix

\[
\begin{pmatrix}
  P & N & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \ddots & N \\
  \vdots & \ddots & P
\end{pmatrix}
\]

[with \( m \) \( P \)'s], \( P \) is the companion matrix of \( p_i(x) \) and \( N \)'s only nonzero entry is a 1 in the upper right corner.

If \( e_i = e_j \), this means \( D_{ij} \) commutes with the "cyclic" matrix \( C_i \), hence by a basic theorem, \( D_{ij} \) is a polynomial in \( C_i \) so "\( D_{ij} " \) contributes \( \deg p_i^{e_i} \) to \( \dim V \), just as desired. If \( e_i \neq e_j \), enlarge \( D_{ij} \) by adding zeros to produce a square matrix \( Y = (D_{ij}) \) or \((0 \ D_{ij}) \). Letting \( e = \max(e_i, e_j) \), we now find \( P_e Y = Y P_e \). Hence \( Y \) is a
polynomial in $P_e$ and so $Y$'s blocks (as induced by $P_e$) necessarily form an upper triangular Toeplitz matrix (equal entries along each diagonal). This readily yields $D_{ij} = (0 E)$ or $\left(\begin{array}{c} E \\ 0 \end{array}\right)$, where $E$ is square of size $m = \min\{e_i, e_j\}$. Again, $P_m E = E P_m$ whence $E$ is a polynomial in $P_m$; thus here "$D_{ij}$" contributes $\deg p_{\min\{e_i, e_j\}}$ to $\dim V$, and we are done.

Finally, the transpose case of our main result. It is true, by way of analogy, that $C D^T = k D C^T$ implies

$$k \text{rank}(C) \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD^T - k BC^T)$$

($l^2 := 1$) but this is not a significant generalization. For the case $k = 1$ is proved by Dade in [2], $k = -1$ is established above, (they could be handled together, of course), $k = 0$ is a straightforward exercise and no other cases are possible. This is easily seen since $(D C^T)^T = CD^T$, and the equation $M^T = k M$ with $M \neq 0$ immediately yields $k^2 = 1$.

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References