\[
\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} \binom{k+1}{j} \binom{n}{j} (j - 1)!
\]

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December 10, 2007

Consider the set of sequences of \( k + 1 \) positive integers, all \( \leq n \), such that the largest entry occurs in the last position (and possibly elsewhere).

On the one hand, \( \sum_{i=1}^{n} i^k \) counts these sequences by last entry \( i \) because there are \( i \) choices for each of the \( k \) preceding positions.

On the other hand, \( \sum_{j=1}^{k+1} \binom{k+1}{j} \binom{n}{j} (j - 1)! \) counts them by number \( j \) of distinct integers occurring in the sequence because each such sequence can be formed (uniquely) as follows. Choose a \( j \)-subset of \([n]\) to serve as the entries appearing in the sequence—\( \binom{n}{j} \) choices. Choose a permutation of this \( j \)-set such that its largest entry occurs last—\( (j - 1)! \) choices—to serve as the permutation obtained from the sequence by erasing all but the last occurrence of each integer appearing in the sequence. Choose a partition of the positions 1, 2, \ldots, \( k + 1 \) into \( j \) nonempty blocks—\( \binom{k+1}{j} \) choices—and order the blocks in increasing order of largest entry. These choices serve to specify the sequence if we place the \( i \)th entry of the permutation into every position in the \( i \)th block. For example with \( n = 7, k = 8, j = 4 \) the \( j \)-set \( \{2, 4, 6, 7\} \), permutation 6 2 4 7, and partition 15-6-37-2489 leads to the sequence (6,7,4,7,6,2,4,7,7), and the process is reversible.