Riordan Numbers Are Differences of Trinomial Coefficients

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The Riordan number $R_n$ [A005043] is the number of Riordan paths of length $n$ where a Riordan path is a Motzkin path [A001006] containing no flatsteps at ground level. Thus $R_4 = 3$ counts

The trinomial coefficient $t(n, k)$, $0 \leq k \leq n$ is the coefficient of $x^k$ in $(x^{-1} + 1 + x)^n$, and $(t(n, 0))_{n \geq 0} = (1, 1, 3, 7, 19, 51, \ldots)$ is the central trinomial coefficient [A002426]. Clearly, $t(n, k)$ counts the set $\mathcal{T}(n, k)$ of lattice paths consisting of upsteps $U = (1, 1)$, flatsteps $F = (1, 0)$ and downsteps $D = (1, -1)$ such that the path (i) contains $n$ steps, and (ii) ends $k$ units above ground level, the horizontal line through its initial point. Thus $\mathcal{T}(n, 0)$ is the set of $n$-step balanced $UFD$-paths and a Motzkin path is a nonnegative balanced $UFD$-path.

Just as the Catalan number is the difference of a central binomial coefficient and its predecessor, the Riordan number is the difference of a central trinomial coefficient and its predecessor:

**Theorem.** $R_n = t(n, 0) - t(n, -1)$.

Since the set $\mathcal{R}_n$ of paths counted by $R_n$ is a subset of $\mathcal{T}(n, 0)$, it suffices to give a bijection from $\mathcal{T}(n, 0) \setminus \mathcal{R}_n$ to $\mathcal{T}(n, -1)$, that is, to the $n$-step $UFD$-paths that end 1 unit below ground level. So suppose given a balanced $UFD$-path of length $n$ that either has
at least one $F$ at ground level or dips below ground level at some point (or both). If the path ends with a $U$ or $F$, rotate this step $45^\circ$, that is, $U \to F$, $F \to D$. This produces, in reversible fashion, all paths in $\mathcal{T}(n,-1)$ that end $F$ or $D$. If the given path ends with a $D$, however, consider its first step and, if it is $U$, let $R$ denote the longest Riordan subpath that terminates the given path. Note that $R$ is necessarily nonempty and not the entire path. Also, the step immediately preceding $R$ in the given path cannot be $D$ (this would violate the maximality of $R$). Thus, in case it ends $D$, the given path has one of the following four mutually exclusive forms ($P$ a $UFD$-subpath), and they are mapped as indicated:

\[
\begin{align*}
DPD & \rightarrow F\overline{P}U \\
FPD & \rightarrow D\overline{P}U \\
UPFR & \rightarrow UPD\overline{R} \\
UPUR & \rightarrow UPF\overline{R}
\end{align*}
\]

Here, $\overline{P}$ is the result of flipping $P$ across a horizontal line. An example for the last form is illustrated.

We leave to the reader the easy verification that this map is invertible and therefore is the desired bijection.