Permutations Avoiding a Nonconsecutive Instance of a 2- or 3-Letter Pattern

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Abstract

We count permutations avoiding a nonconsecutive instance of a two- or three-letter pattern, that is, the pattern may occur but only as consecutive entries in the permutation. Two-letter patterns give rise to the Fibonacci numbers. The counting sequences for the two representative three-letter patterns, 321 and 132, have respective generating functions \((1 + x^2)(C(x) - 1)/(1 + x + x^2 - xC(x))\) and \(C(x + x^3)\) where \(C(x)\) is the generating function for the Catalan numbers.

1. Introduction
There is a large literature on pattern avoidance in permutations and words. Problems treated include counting permutations avoiding a pattern or set of patterns, or containing patterns a specified number of times. Or a pattern may be allowed but only as part of a larger pattern. Pattern occurrences may be unrestricted (the classical case), or required to be consecutive (i.e., contiguous entries in the permutation), or a mixture of the two. Here we consider the following variation. Given a pattern \(\pi\), how many permutations on \([n]\) avoid nonconsecutive instances of \(\pi\), that is, the pattern \(\pi\) may occur but only as consecutive entries in the permutation. In other words, all occurrences of \(\pi\) are required to be of the consecutive type. As usual, for \(\pi = (\pi_1, \pi_2, \ldots, \pi_k)\) a permutation on \([k]\), an instance of \(\pi\) in a permutation \((a_1, a_2, \ldots, a_n)\) is a subpermutation \((a_{i_1}, a_{i_2}, \ldots, a_{i_k})\), \(1 \leq i_1 < \ldots < i_k \leq n\), whose reduced form (replace smallest entry by 1, next smallest by 2, and so on) is \(\pi\). In the present paper we deal with patterns of length \(\leq 3\). The operations reverse and complement
$(a_i \rightarrow n + 1 - a_i)$ on permutations and patterns preserve this type of pattern avoidance; so we need only consider representative patterns in the orbits under the action of the group they generate. For patterns of length $\leq 3$, this reduces the problem (as usual) to the patterns 21, 321 and 132. The following sections treat them one at a time.

2. The 21 pattern Avoiding a nonconsecutive 21 means that every inversion arises from a contiguous pair of entries. Such permutations on $[n]$ are all obtained from the identity permutation $(1, 2, \ldots, n)$ by selecting a set of non-overlapping pairs of consecutive integers in $[n]$ and, for each pair, switching the entries in those 2 positions. The first entries of these pairs form a scattered subset of $[n-1]$, that is, a subset for which distinct elements differ by at least 2. It is well known that such subsets are counted by the Fibonacci number $F_{n+1}$. Hence we have

**Theorem 1.** The number of permutations on $[n]$ that avoid a nonconsecutive 21 pattern is $F_{n+1}$.

3. The 321 pattern Let us introduce some notation:

- $A_n$ is the full set of permutations on $[n]$ that avoid a nonconsecutive 321.
- $B_n, D_n$ are, respectively, the permutations in $A_n$ that start with a 321 in the first 3 positions, and those that don’t. Thus $|A_n| = |B_n| + |D_n|$.
- $C_n$ is the set of permutations in $A_n$ that contain no 321 at all, the set of so-called 321-avoiding permutations. It is well known [1] that $|C_n| = C_n$, the $n$th Catalan number.
- $A_{n,k}$, $(1 \leq k \leq n - 2)$ is the set of permutations in $A_n$ that contain one or more (necessarily consecutive) 321s, the first of which starts at position $k$. Thus $|A_n| = C_n + \sum_{k=1}^{n-2} |A_{n,k}|$.

First, observe that a permutation in $B_n$ ($n \geq 3$) must have 2 as its second entry and 1 as its third entry (else an offending pattern would be present). Deleting these entries and subtracting 2 from all other entries is a bijection to $D_{n-2}$. Hence $|B_n| = |D_{n-2}|$. Next,
given $\rho \in \mathcal{A}_{n,k}$ $(2 \leq k \leq n - 2)$, we can form two new permutations $\sigma, \tau$ as follows. Take the first $k - 1$ entries and the entry in position $k + 2$ (necessarily comprising the first $k$ positive integers) to form $\sigma$. Delete the first $k - 1$ entries and reduce (replace smallest entry by 1, next smallest by 2, and so on) to form $\tau$. We leave the reader to verify that this is a bijection $\mathcal{A}_{n,k} \rightarrow C_k \times \mathcal{B}_{n-k+1}$. Hence $|\mathcal{A}_{n,k}| = C_k |\mathcal{B}_{n-k+1}| = C_k |D_{n-k-1}|$.

Now, with a lowercase letter $a_n$ denoting the size of $\mathcal{A}_n$ and so on, we have just shown that $a_n = C_n + \sum_{k=1}^{n-2} C_k d_{n-k-1}$ (since $C_1 = 1$). But also $a_n = b_n + d_n$ and $b_n = d_{n-2}$. Eliminating $a_n$ and $b_n$ and reindexing yields

$$d_n = C_n + \sum_{k=1}^{n-3} C_{k+1} d_{n-2-k},$$

a recurrence for $d_n$ that is valid for $n \geq 1$ and hence determines $d_n$ (no initial condition necessary). Since the sum is a convolution, this recurrence routinely yields the generating function

$$D(x) = \frac{C^*(x)}{1 + x^2 - x C^*(x)},$$

where $D(x) := \sum_{n \geq 1} d_n x^n$ and $C^*(x) := \sum_{n \geq 1} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} - 1$ (the superscript $^*$ indicating that $C_0$ is omitted from the sum). The formula for $a_n$ then yields

**Theorem 2.** The number $a_n$ of permutations on $[n]$ that avoid a nonconsecutive 321 pattern has generating function

$$\sum_{n \geq 1} a_n x^n = \frac{C^*(x)}{1 - \frac{x}{1 + x} C^*(x)}.$$

The sequence $(a_n)_{n \geq 1}$ begins 1, 2, 6, 18, 56, 182, 607, 2064, \ldots.

3. **The 132 pattern** First, observe that 132-avoiding permutations are characterized by the property that, for each entry $a$, the set of succeeding entries that are $< a$ form an initial segment of the positive integers. Also, Simion and Schmidt [2] gave a bijection (see [3] for an equivalent description) from 132-avoiding to 123-avoiding permutations, the latter corresponding to 321-avoiding under reversal, and so 132-avoiding permutations are also counted by the Catalan numbers.

Now let $\mathcal{E}_n$ denote the set of permutations on $[n]$ that avoid a nonconsecutive 132 and $\mathcal{E}_{n,k}$ the permutations in $\mathcal{E}_n$ with $k$ (necessarily consecutive) 132s. Consider a permutation
in $\mathcal{E}_{n,k}$. First, record the positions $i_1, i_2, \ldots, i_k$ of the middle entries of its 132 patterns—
they form a subset of the interval $[2, n - 1]$ whose entries all differ by at least 3 because, as
is easily checked, the 132 patterns cannot overlap. Then delete the first and last entry of
each 132 pattern and reduce. The result is a 132-avoiding permutation on $[n - 2k]$. Thus
each permutation in $\mathcal{E}_{n,k}$ yields a set $\{i_1, i_2, \ldots, i_k\} \subset [2, n - 1]$ with entries differing by
at least 3 together with a 132-avoiding permutation on $[n - 2k]$.

Conversely, given any such set and permutation, we can construct a permutation
in $\mathcal{E}_{n,k}$ that yields them. Take, for example, $n = 10$ and $k = 2$ and suppose given
$\{i_1, i_2, \ldots, i_k\} = \{4, 8\}$ and 132-avoiding permutation $(6, 5, 3, 4, 2, 1)$. Place blanks in
the positions immediately neighboring each $i_j$ and place the permutation entries in the
remaining positions:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
6 & 5 & 3 & 4 & 2 & 1
\end{array}
\]

Leave intact the entries after the entry in position $i_1$ and smaller than it (as noted above,
these entries form an initial segment $1, 2, \ldots, j$ of the positive integers), place $j + 1$ in
position $i_1 - 1$, $j + 2$ in position $i_1 + 1$, and increase all other entries by 2:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
8 & 7 & 3 & 5 & 4 & 6 & 2 & 1
\end{array}
\]

Proceed likewise left to right to fill in the remaining pairs of blanks. It is easy to check
that the result, here $(10, 9, 5, 7, 6, 8, 2, 4, 3, 1)$, is a permutation in $\mathcal{E}_{n,k}$ that yields the given
$\{i_1, i_2, \ldots, i_k\}$ and 132-avoiding permutation under the process of the previous paragraph.

There are $\binom{n - 2k}{k}$ such sets $\{i_1, i_2, \ldots, i_k\}$ and $C_{n-2k}$ 132-avoiding permutations on
$[n - 2k]$ and so $|\mathcal{E}_{n,k}| = \binom{n - 2k}{k} C_{n-2k}$. Summing over $k$ and taking into account the
permutations that avoid 132 altogether, we have

**Theorem 3.** The number of permutations on $[n]$ that avoid a nonconsecutive 132 pattern
is

\[
\left\lfloor \frac{n}{3} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n}{3} \right\rfloor} \binom{n - 2k}{k} C_{n-2k}.
\]
Routine manipulations lead to a succinct generating function:

\[
\sum_{n \geq 0} |\xi_n| x^n = \sum_{n, k \geq 0} \binom{n - 2k}{k} C_{n-2k} x^n \\
= \sum_{k \geq 0} \left( \sum_{n \geq 3k} \binom{n - 2k}{k} C_{n-2k} x^{n-3k} \right) x^{3k} \\
= \sum_{k \geq 0} \left( \sum_{n \geq k} \binom{n}{k} C_n x^{n-k} \right) x^{3k} \\
= \sum_{k \geq 0} \frac{C^k(x)}{k!} x^{3k} \\
= C(x + x^3),
\]

the last equality by Taylor’s theorem, where \( C(x) = \frac{1-\sqrt{1-4x}}{2x} \) is the generating function for the Catalan numbers. So we have

**Corollary.** The generating function for permutations that avoid a nonconsecutive 132 pattern is

\[
\frac{1 - \sqrt{1 - 4x - 4x^3}}{2(x + x^3)}.
\]

The counting sequence \((n \geq 0)\) begins 1, 1, 2, 6, 18, 57, 190, 654, 2306, \ldots .

**Added in Proof** These results have previously been obtained in a wider context by Anders Claesson [4].

**References**

