SROAD Designs for $k \geq 4$

by
Norman R. Draper
Friedrich Pukelsheim
and
Lisa H. Ying
SROAD Designs for $k \geq 4$

Norman R. Draper, Friedrich Pukelsheim, and Lisa H. Ying

ABSTRACT

Slope rotatability over all directions (SROAD) is a useful concept when the slope of a second order response is to be studied. SROAD designs ensure that knowledge of the slope is acquired symmetrically, whatever direction later becomes of more interest as the data are analyzed. In a prior report, we explored designs for $k = 2$ and $3$ dimensions which do not have the full symmetries of second order designs but which nevertheless possess the SROAD property. Here we discuss designs in higher dimensions. (The introductory sections 1 and 2 are essentially identical to those of the prior report.)

Key words: Moment matrices, Response surface, Slope rotatability.
1 Introduction

In some response surface applications, attention focuses on the estimation of differences in response, or slopes rather than the absolute value of the response variable. It is then natural to consider the variance measure for the slope of the fitted surface at any given point. On the assumption that equal information in all directions about the design origin was important, Hader and Park (1978) introduced the idea of slope rotatability and discussed it in the context of central composite designs. Later, Park (1987) introduced second order slope rotatability over all directions (hereafter simply denoted by SROAD), and gave necessary and sufficient conditions for a design to have this property based on the precision matrix. Only a few simple types of SROAD designs have been discussed in the literature. The purpose of this paper is to investigate the moment structures of SROAD designs in detail and so to find alternative SROAD designs.

2 Slope Rotatability Over All Directions

Suppose the response variable $y$ satisfies a functional relationship in $k$ predictor variables, $x_1, x_2, ..., x_k$, of general form: $y_i = \eta(x_i) + \epsilon_i$, where, $y_i$ is the observed response taken at a selected combination, $x_i = (x_{i1}, x_{i2}, ..., x_{ik})'$ of the predictor variables $x = (x_1, x_2, ..., x_k)'$, and $i = 1, 2, ..., N$. The $\epsilon_i$ are assumed to be uncorrelated random errors with zero means and constant variance, $\sigma^2$. We assume that $\eta$ can be represented adequately in a restricted region of interest by a second order polynomial

$$\eta(x) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \sum_{j \geq i}^{k} \beta_{ij} x_i x_j = z_x' \beta,$$  \hspace{1cm} \text{(1)}$$

where $z'_x = (1, x_1, x_2, ..., x_k, x_1^2, x_2^2, ..., x_k^2, x_1 x_2, ..., x_{k-1} x_k)$ and $\beta$ is the vector of correspondingly subscripted coefficients. The least squares estimator of $\beta$ is $b = (X'X)^{-1}X'y$, where $X$ is the matrix of values of $z'_x$ taken at the $N$ design points, and $y$ is the $N \times 1$ vector of $y$ observations. It is traditional to call $N^{-1}XX'$ the moment matrix and its inverse $N(X'X)^{-1}$ the precision matrix.
The first derivatives (slopes) of $\eta(x)$ at a general point $u = (u_1, ..., u_k)'$ in the x-space are

$$\left. \frac{\partial \hat{y}}{\partial x_i} \right|_u = b_i + 2b_{ii}u_i + \sum_{j \neq i} b_{ij}u_j.$$  

(2)

Denote the estimated slope vector by

$$g(u) = \left( \frac{\partial \hat{y}}{\partial x_1}, \frac{\partial \hat{y}}{\partial x_2}, ..., \frac{\partial \hat{y}}{\partial x_k} \right)' = Ab,$$

(3)

say. The estimated derivative at any point $u$ in the direction specified by a $k \times 1$ vector of direction cosines $c' = (c_1, c_2, ..., c_k)$, is thus $c'g(u)$, where $c'c = 1$. The variance of this is

$$V_c(u) = Var[c'g(u)] = \sigma^2 c' A(X'X)^{-1} A'c.$$  

(4)

The (integrated) averaged value of $V_c(u)$ over all possible directions, namely the \textit{averaged slope variance} is

$$\bar{V}(u) = \frac{\sigma^2}{k} tr[A(X'X)^{-1} A'];$$

(5)

see Park (1987, pp. 450-451). $\bar{V}(u)$ is a function of the point $u$, through $A$, and also a function of the design, through $X$. A $k$-dimensional design is said to be a slope rotatable over all directions (SROAD) if $\bar{V}(u)$ at the point $u$ depends only on the distance of the point $u$ from the design origin (Park, 1987). The necessary and sufficient conditions for a second-order design to be SROAD are (Park, 1987):

1. $2Cov(b_i, b_{ii}) + \sum_{j=1,j \neq i}^k Cov(b_j, b_{ij}) = 0,$ for all $i$.
2. $2[Cov(b_{ii}, b_{ij}) + Cov(b_{jj}, b_{ij})] + \sum_{t=1,t \neq i,j}^k Cov(b_{it}, b_{jt}) = 0,$
   for any $(i, j)$, when $i \neq j$.
3. $4Var(b_{ii}) + \sum_{j=1,j \neq i}^k Var(b_{ij}),$ equal for all $i$.

Some designs given by Park (1987, p452) followed from his
"Corollary 1: If the following moment conditions are satisfied, the design is slope-rotatable over all directions.

1. All odd-order moments are 0 (i.e., \([i] = [ij] = [iij] = [iii] = [iiij] = \ldots = 0\)).
2. \([ii]\) are equal for all \(i\).
3. \([iiii]\) are equal for all \(i\).
4. \([iijj]\) are equal for all \(i \neq j\)."

The quantities in square brackets denote the moments of the design. For example, 
\[N^{-1} \sum_{u=1}^{N} x_{i u} = [i], \quad N^{-1} \sum_{u=1}^{N} x_{i u}^2 x_{j u} = [iij]\] and so on. If any subscript \(i, j, \ldots\), appears an odd number of times, the moment is odd. Otherwise, it is even.

This corollary is framed using moment conditions for specific types of designs that are known to be rotatable (that is, \(Var(\hat{y}(u)) = \text{function only of} \ u^t u\), see Box and Hunter, 1957, p. 205) or slope rotatable (that is, \(Var(\frac{\partial y}{\partial x_i} | u) = \text{function only of} \ u^t u\) and \(Var(\frac{\partial y}{\partial x_i} | u) = \text{the same for all} \ i = 1, \ldots, k\), see Hader and Park, 1978, p. 414). The corollary thus implies that we have the following relationship among designs:

\[\text{rotatable} \subset \text{slope-rotatable} \subset \text{slope-rotatable over all directions}.\]

This means that the class of SROAD designs contains the class of slope rotatable designs which itself contains the class of rotatable designs. Thus, there must be a much wider choice of SROAD designs. In particular, we have found some with unbalanced moment structures, and some with unbalanced point arrangements. Odd order moments of an SROAD design could be non-zero. The types of SROAD designs vary from dimension to dimension. We discuss \(k \geq 4\) dimensions here, giving examples of designs that are SROAD but do not belong to the classes of rotatable designs and slope-rotatable designs. None of these have been given by previous authors.
3 Four Dimensional SROAD Designs

In four dimensions, we have found some new types of moment structures for SROAD designs different from those in two and three dimensions. The results are summarized in the following lemma. The lemma is obtained by finding a class of designs that satisfies the conditions on the precision matrix and inverting that matrix symbolically via MAPLE to obtain conditions on the moment matrix.

**Lemma 1** Suppose the moments of a design satisfy:

1. All odd-order moments except [123] and [1234] are 0.

2. $[33] = [22] = [11] = \lambda_{11}$.

3. $[3333] = [2222] = [1111] = \lambda_{1111}$.

4. $[3344] = [1122] = \lambda_{1122}$.

5. $[2244] = [1133] = \lambda_{1133}$.

6. $[2233] = [1144] = \lambda_{1144}$.

Then the design is SROAD if and only if its moments satisfy the following equations.

$$\lambda_{123} \lambda_{1234} \sum_{s=2}^{4} \left( \lambda_{11s}^2 \lambda_{11} - \lambda_{11s} \lambda_{123}^2 - \lambda_{1234}^2 \lambda_{11} \right)^{-1} = 0,$$

$$4 (\lambda_{1144} - \lambda_{11jj}) \left\{ (\lambda_{11jj} + \lambda_{1144}) \left( \lambda_{11}^2 - \lambda_{14}^2 - \lambda_{1111} + \lambda_{4444} \right) \right. $$

$$+ 2 \lambda_{11} \lambda_{44} (\lambda_{1111} + \lambda_{11ii}) - 2 \lambda_{11}^2 (\lambda_{4444} + \lambda_{11ii}) \} / \varphi$$

$$ + \lambda_{123}^2 \left\{ \left( \lambda_{11jj}^2 \lambda_{11} - \lambda_{11jj} \lambda_{123}^2 - \lambda_{1234}^2 \lambda_{11} \right)^{-1} - \left( \lambda_{1144}^2 \lambda_{11} - \lambda_{1144} \lambda_{123}^2 - \lambda_{1234}^2 \lambda_{11} \right)^{-1} \right\} = 0,$$

for $i, j = 2, 3$, and $i \neq j$.  

(6)  

(7)
(There are two equations in (7), \((i,j) = (2,3)\) and \((i,j) = (3,2)\).)

\[
4 \left( \lambda_{1144} - \lambda_{1111} \right) \left\{ \left( \lambda_{1111} + \lambda_{1144} \right) \left( \lambda_{11}^2 - \lambda_{44}^2 - \lambda_{1111} + \lambda_{4444} \right) \\
+ 2 \lambda_{11} \lambda_{44} \left( \lambda_{1122} + \lambda_{1133} \right) + 2 \lambda_{11}^2 \left( \lambda_{1111} - \lambda_{4444} - \lambda_{1122} - \lambda_{1133} \right) \right\} / \varphi \\
+ \lambda_{123}^2 \sum_{s=2}^{3} \left( \lambda_{11ss} \lambda_{11} - \lambda_{11ss} \lambda_{123} - \lambda_{1234} \lambda_{11} \right)^{-1} = 0,
\]

(8)

where

\[
\varphi = \lambda_{1111} \left( \lambda_{1111} + \lambda_{4444} - \lambda_{44}^2 \right) \sum_{s=2}^{4} \lambda_{11ss}^2 + \lambda_{1111}^3 \left( \lambda_{44}^2 - \lambda_{4444} \right) \\
- \left( \lambda_{1122} - \lambda_{1133} - \lambda_{1144}^2 \right)^2 - \lambda_{11} \lambda_{4444} \left( \lambda_{1122} - \lambda_{1133} - \lambda_{1144} \right)^2 \\
+ \lambda_{1122} \lambda_{1133} \lambda_{1144} \left\{ 2 \left( \lambda_{44}^2 - \lambda_{4444} \right) + 6 \left( \lambda_{11}^2 - \lambda_{1111} \right) \right\} \\
+ 4 \lambda_{1133} \lambda_{1144}^2 - 2 \lambda_{11} \lambda_{44} \sum_{s=2}^{4} \lambda_{11ss} \sum_{t=1}^{4} (-1)^{t=s} \lambda_{11tt} \\
- 2 \lambda_{1111} \sum_{s \neq t \neq 1} \lambda_{11ss} \lambda_{11tt} - 2 \lambda_{11}^2 \left\{ \lambda_{1111} \sum_{s=2}^{4} \lambda_{11ss}^2 \\
+ \sum_{s=2}^{4} \lambda_{11ss} \sum_{t=2}^{4} (-1)^{t=s} \lambda_{11tt} - \lambda_{4444} \left( 3 \lambda_{1111}^2 / 2 + 2 \lambda_{1133} \lambda_{1144} \right) \\
+ \lambda_{1111} \left( \lambda_{4444} \sum_{s=2}^{4} \lambda_{11ss} - \lambda_{1122} \left( \lambda_{1133} + \lambda_{1144} \right) - \lambda_{1133} \lambda_{1144} \right) \right\}
\]

and \(I_{(t=s)}\) is an indicator function such that

\[
I_{(t=s)} = \begin{cases} 
1 & \text{if } t = s \\
0 & \text{if } t \neq s
\end{cases}
\]

This lemma shows that the even moments \([ijij]\) of a four-dimensional SROAD design would be separated into several groups and other same order even moments would not all be equal. Some special cases of this lemma are shown in the examples below.

Case 1. A simple subcase of Lemma 1 is that a design is SROAD if, additionally, \([123]=0\), \([ii]\) are all equal, and \([iiii]\) are all equal. Equations (6) – (8) are then all satisfied.
Example 1. A design with the following points is SROAD, for any $a, b, \alpha$ and $n_0$:

1. $\frac{1}{2}S(a, a, a, a)$ with $I = -1234$ (8 points),
2. $\frac{1}{2}S(b, b, b, b)$ with $I = 1234$ (8 points),
3. $S(\alpha, 0, 0, 0)$ (8 points).
4. $n_0$ center points.

The notation $S(a, a, a, a)$ denotes the 16 factorial points $(\pm a, \pm a, \pm a, \pm a)$. Set (1) is thus a $2_{III}^{4-1}$ design. "$S(\alpha, 0, 0, 0)$" denotes the $2k$ axial points $(\pm \alpha, 0, 0, 0), \ldots, (0, 0, 0, \pm \alpha)$. For this example, $N = 24 + n_0, [ii] N = 8(a^2 + b^2) + 2\alpha^2$, and $[iii] N = 8(a^4 + b^4) + 2\alpha^4$, for all $i$; $[iijj] N = 8(a^4 + b^4)$ for all $i \neq j$; and $[1234] N = 8(a^4 - b^4)$. When $a = b = 1$, this design becomes a $2^4$ center composite design.

Example 2. Draper and McGregor (1967) gave an infinite class of second order rotatable designs which combined point sets with non-equal fourth moment [iijj] for four dimensions. We can use part of such a design, and add a set of axial points. Our selected design contains the following points:

1. 32 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u})$ with coordinates chosen from
   
   $$(\pm x, \pm y, \pm z, \pm w)$$

   $$(\pm y, \pm z, \pm w, \pm x)$$

   $$(\pm z, \pm w, \pm x, \pm y)$$

   $$(\pm w, \pm x, \pm y, \pm z)$$

   where only combinations of signs are used which make $x_{1u}x_{2u}x_{3u}x_{4u} = xyzw$. An alternative description is that, it is generated by the cyclic permutation $\{(x, y, z, w) \Rightarrow$
\[(y, z, w, x) \Rightarrow (z, w, x, y) \Rightarrow (w, x, y, z)\} \text{ of the following matrix } D_1.

\[
D_1 = \begin{pmatrix}
-x & -y & -z & -w \\
x & y & -z & -w \\
-x & y & z & -w \\
x & -y & z & -w \\
-x & -y & z & w \\
x & y & -z & w \\
-x & y & -z & w \\
x & y & z & w
\end{pmatrix}.
\]

2. 8 axial points with distance \(\alpha\).

3. \(n_0\) center points.

For this design, \(N = 40 + n_0\), [ii] \(N = 8(x^2 + y^2 + z^2 + w^2) + 2\alpha^2\), [iii] \(N = 8(x^4 + y^4 + z^4 + w^4) + 2\alpha^4\), [ij] \([ijj]\) are separated into two groups: [iijj] \(N = 16(x^2z^2 + y^2w^2)\) for \((i, j) = (1, 3), (2, 4)\), [iijjj] \(N = 8(x^2 + z^2)(y^2 + w^2)\) for \((i, j) = (1, 2), (2, 3), (3, 4), (4, 1)\), [1234] \(N = 32xyzw\). The design is SROAD for any values of \(x, y, z, w, \alpha\), and any \(n_0\) (as is the full set of Draper and McGregor’s design).

**Example 3.** Suppose a design contains the following points:

1. 32 points \((x_{1u}, x_{2u}, x_{3u}, x_{4u})\) with coordinates

\[
(\pm x, \pm y, \pm z, \pm w) \\
(\pm y, \pm z, \pm w, \pm x) \\
(\pm z, \pm w, \pm x, \pm y) \\
(\pm w, \pm x, \pm y, \pm z)
\]

where only combinations of signs are used which make \(x_{1u}x_{2u}x_{3u}x_{4u} = xyzw\).

2. 32 points \((x_{1u}, x_{2u}, x_{3u}, x_{4u})\) like (1) with coordinate values \(a, b, c, d\) replacing \((x, y, z, w)\).
3. 8 axial points with distance $\alpha$.

4. $n_0$ center points.

In this case, $N = 72 + n_0$, $[i]N = 8(x^2 + y^2 + z^2 + w^2 + a^2 + b^2 + c^2 + d^2) + 2\alpha^2$,

$[iii]N = 8(x^4 + y^4 + z^4 + w^4 + a^4 + b^4 + c^4 + d^4) + 2\alpha^4$, $[ijj]$ are separated into three groups: $[ijj]N = 16(x^2z^2 + y^2w^2) + 8(a^2 + d^2)(b^2 + c^2)$ for $(i, j) = (1, 3), (2, 4)$,

$[ijj]N = 16(a^2d^2 + b^2c^2) + 8(x^2 + z^2)(y^2 + w^2)$ for $(i, j) = (1, 4), (2, 3)$, $[ijj]N = 8[(x^2 + z^2)(y^2 + w^2) + (a^2 + d^2)(b^2 + c^2)]$ for $(i, j) = (1, 2), (3, 4)$, $[1234]N = 32(xyzw + abcd)$.

The design is SROAD for any values of $x, y, z, w, a, b, c, d, \alpha$, and any $n_0$.

**Case 2.** Another subcase of Lemma 1 occurs when, additionally, $[1234] = 0$ and the $[ijj]$ are all equal. In this case, Equations (6) and (7) are satisfied and (8) reduces to

$$4 \left\{ (\lambda_{1122} + \lambda_{1111})(\lambda_{1111} - \lambda_{4444} + \lambda_{44}^2) + 2 \lambda_{11} (\lambda_{11} \lambda_{4444} - 2 \lambda_{44} \lambda_{1122}) \\
+ 3 \lambda_{11}^2 (\lambda_{1122} - \lambda_{1111}) \right\} / \varphi + 2 \lambda_{123}^2 / (\lambda_{1122}\lambda_{11} - \lambda_{1122} \lambda_{123}^2) = 0,$$

where $\lambda_{1122}$ denotes $[ijj]$ and

$$\varphi = (\lambda_{1111} - \lambda_{1122}) \left\{ \lambda_{1122} [3 (\lambda_{1122} - 2 \lambda_{11} \lambda_{44}) + 2 (\lambda_{44}^2 - \lambda_{4444})] \\
+ 3 \lambda_{11} \lambda_{4444} + \lambda_{1111} (\lambda_{44}^2 - \lambda_{4444}) \right\}.$$
Example 4. Consider the design containing \{a \, 2^{4-1} \text{ with } I = 123 \} plus \{ \text{star points with distance } \alpha \text{ for factor 1, 2, 3, and } \gamma \text{ for factor 4} \} \text{ plus } \{n_0 \text{ center points} \}$:

\[
D = \begin{pmatrix}
-1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 \\
-\alpha & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & -\alpha & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & -\gamma \\
0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The design is essentially an extension into $k = 4$ dimensions of Hartley's $k = 3$ design. (See Ying, Pukelsheim and Draper (1994).) For this design, $[11] = [22] = [33] = 2(4 + \alpha^2)$, $[44] = 2(4 + \gamma^2)$; $[1111] = [2222] = [3333] = 2(4 + \alpha^4)$, $[4444] = 2(4 + \gamma^4)$; $[iijj] = [123] = 8$, for all $i \neq j$. The design is SROAD if and only if

\[
\left\{ \alpha^6(n_0 + 8) + 2\alpha^4(n_0 - 14) + 4\alpha^2(3n_0 + 2) + 16(n_0 + 6) \right\} \gamma^4 - 16\alpha^2 \\
\times (\alpha^4 - \alpha^2 - 4)\gamma^2 - 2\alpha^3 \left\{ (n_0 + 10)(\alpha^4 + 8) - 2\alpha^2(n_0 + 26) \right\} = 0. \tag{10}
\]
Values of $\gamma$ which make the design SROAD versus $\alpha$ for selected $n_0$ are shown in Table 1 and Figure 1. As was the case for Hartley designs, rotation of the x-axes can make all the third and fourth order odd moments of the above SROAD designs non-zero (after rotation).

Table 1: Example 4: the values of $\gamma$ for selected $\alpha$, $n_0$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.806</td>
</tr>
<tr>
<td>1.25</td>
<td>0.922</td>
</tr>
<tr>
<td>1.50</td>
<td>1.081</td>
</tr>
<tr>
<td>1.75</td>
<td>1.391</td>
</tr>
<tr>
<td>2.00</td>
<td>1.714</td>
</tr>
<tr>
<td>2.25</td>
<td>1.932</td>
</tr>
<tr>
<td>2.50</td>
<td>2.080</td>
</tr>
</tbody>
</table>

Figure 1: Plot of the values of $\gamma$ versus $\alpha$ for Example 4.
4 k-Dimensional SROAD Designs, $k \geq 5$

Some results in three and four dimensions can be extended to higher dimensions.

**Case 1.** For dimension $k = 3p$, where $p$ is an integer. If the following moment conditions are satisfied, the design is SROAD.

1. All odd-order moments with order less than 5 except $[(3j - 2)(3j - 1)(3j)]$, $j = 1, \ldots, p$, are 0.
2. $[(3j - 2)(3j - 1)(3j)]$ are equal for all $j = 1, \ldots, p$.
3. $[ii]$ are equal for all $i$.
4. $[iiii]$ are equal for all $i$.
5. $[iiij]$ are equal for all $i \neq j$.

The proof is omitted. The $k = 3p$ Hartley designs which consist of a $2^{k-p}$ design with $I = 123 = 456 = \ldots = (3p - 2)(3p - 1)3p$, plus $2k$ star points at distance $\alpha$, plus $n_0$ center points, satisfy the above moment conditions. The Hartley designs can also be regarded as members of a larger specific general class involving $2^{k-p}$ fractional factorial designs.

**Case 2.** Consider designs that contain:

1. A $2^{k-p}$ design with $p = \lfloor \frac{k}{3} \rfloor$ (the integer part of $k/3$) and $I = 123 = 456 = \ldots = (3p - 2)(3p - 1)3p$, where $k \geq 3$.
2. Star points with distance $\alpha$ for factor $1, \ldots, 3p$ and $\gamma$ for factor $3p + 1, \ldots, k$.
3. $n_0$ center points.

Then

(a) If $k = 3p$, the design is a Hartley design and is SROAD for any $\alpha$. 


(b) For \( k = 3p + 1 \), the design is SROAD if \( \alpha, \gamma \) and \( n_0 \) satisfy:

\[
\left\{ \alpha^6(n_0 + 2^{k-p}) + 2 \alpha^4[n_0 - (k - 2) 2^{k-p} + 2] + 2^{k-p-1}\alpha^2 \times \\
[(k - 1) n_0 + 18 p^2 - 24 p + 8] + 2^{k-p}(k - 2) (n_0 + 6 p) \right\} \gamma^4 \\
-2^{k-p+1}\alpha^2 \left\{ \alpha^4 - (k - 3) \alpha^2 - 2 (k - 2) \right\} \gamma^2 \\
-2 \alpha^4 \left\{ \alpha^4(n_0 + 2^{k-p} + 2) - 2^{k-p-2}\alpha^2(n_0 + 24 p + 2) \\
+2^{k-p-1}(k - 2) [n_0 + 2 (k + 1)] \right\} = 0.
\]

(11)

**Example 5** \((k = 7)\). Consider the \( 2^{7-2}_{III} \) design with \( I = 123 = 456 \). Values of \( \gamma \) for selected \( \alpha, n_0 \) are listed in Table 2. The plot of these values would be similar to Figure 1, except that the curves diverge later at about \( \alpha = 2.15 \) (rather than at 1.5).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( n_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1.00</td>
<td>0.865</td>
</tr>
<tr>
<td>1.25</td>
<td>1.014</td>
</tr>
<tr>
<td>1.50</td>
<td>1.134</td>
</tr>
<tr>
<td>1.75</td>
<td>1.228</td>
</tr>
<tr>
<td>2.00</td>
<td>1.315</td>
</tr>
<tr>
<td>2.25</td>
<td>1.474</td>
</tr>
<tr>
<td>2.50</td>
<td>1.845</td>
</tr>
</tbody>
</table>

(c) If \( k = 3p + 2 \), the design is SROAD when \( \alpha, \gamma \) and \( n_0 \) satisfy the following equation.

\[
\left\{ \alpha^6(n_0 + 2^{k-p}) + 2 \alpha^4[n_0 - (k - 3) 2^{k-p} + 2] + 2^{k-p-1}\alpha^2 \times \\
[(k - 2) n_0 + 18 p^2 - 24 p + 8] + 2^{k-p}(k - 3) (n_0 + 6 p) \right\} \gamma^8 \\
-2^{k-p+2}\alpha^2 \left\{ \alpha^4 - (k - 4) \alpha^2 - 2 (k - 3) \right\} \gamma^6 \\
-2 \alpha^4 \left\{ \alpha^4(n_0 + 2^{k-p} + 2) - 2^{k-p-1}\alpha^2(n_0 + 12 p + 4) \\
+2^{k-p-1}[(k - 4) n_0 + 6 p(k - 1) - 12)] \right\} \gamma^4 \\
+2^{k-p+2}\alpha^6(\alpha^2 - 3 p) \gamma^2 - 2^{k-p}\alpha^8(n_0 + 4) = 0.
\]

(12)
Example 6 ($k = 5$). Consider the $2^5_{III}$ design with $I = 123$. We see that we again have an extension of the $k = 3$ Hartley design, this time into five dimensions rather than four as previously. Equation 12 becomes:

$$\left\{a^6(n_0 + 16) + 2a^4(n_0 - 30) + 8a^2(3n_0 + 2) + 32(n_0 + 6)\right\} \gamma^8$$

$$-64a^2(a^4 - a^2 - 4)\gamma^6 - 2a^4\{a^4(n_0 + 18) - 8a^2(n_0 + 16)$$

$$+8(n_0 + 12)\}\gamma^4 + 64a^6(a^2 - 3)\gamma^2 - 16a^8(n_0 + 4) = 0. \quad (13)$$

Some values of $\gamma$ which make the design SROAD for selected $\alpha$, $n_0$ are listed in Table 3. The plot of these values resembles Figure 1 but the curves diverge now at $\alpha = 1.65$.

Table 3: Values of $\gamma$ for selected $\alpha$, $n_0$ ($k = 5$, $p = 1$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n_0$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1.00</td>
<td>0.851</td>
<td>0.852</td>
</tr>
<tr>
<td>1.25</td>
<td>1.001</td>
<td>1.002</td>
</tr>
<tr>
<td>1.50</td>
<td>1.137</td>
<td>1.139</td>
</tr>
<tr>
<td>1.75</td>
<td>1.280</td>
<td>1.278</td>
</tr>
<tr>
<td>2.00</td>
<td>1.472</td>
<td>1.445</td>
</tr>
<tr>
<td>2.25</td>
<td>1.807</td>
<td>1.673</td>
</tr>
<tr>
<td>2.50</td>
<td>2.085</td>
<td>1.921</td>
</tr>
</tbody>
</table>

Proofs of results (a), (b), and (c) are omitted.
Case 3. SROAD designs with $[iijj]$ not all equal. In $k$-dimensions ($k \geq 4$), a design is SROAD if the following moment conditions are satisfied.

1. All odd-order moments up to and including four are 0.
2. $[ii]$ are equal for all $i$, $[iiii]$ are equal for all $i$.
3. $[iijj]$ $i \neq j$ are separated to $n$ groups ($n < k$). Each group contains $kp$ or $kp/2$ of $[iijj]$ with equal value $\tau_{ui}$, $i = 1, ..., n$, where $p$ is an integer, and each number of 1, ..., $k$ appears exactly the same times in the $[iijj]$ of each group.

The moment matrix $(X'X)/N$ of this type of design and its inverse have exactly the same structure. That is, $\text{Var}(b_i)$ are equal for all $i$, $\text{Var}(b_{ii})$ are equal for all $i$, $\text{Var}(b_{ij})$ are equal for all $i, j$ in the same group, $\text{Cov}(b_{ii}, b_{ii})$ are equal for all $i, \text{Cov}(b_{ii}, b_{jj})$ are equal for all $i, j$ in the same group, $\text{Cov}(b_{ij}, b_{st})$ are equal for all $i \neq j \neq s \neq t$, other covariances are 0.

Additionally, for $4p$ dimensions, we can have $[1234] = [5678] = ... = [(4p-3)(4p-2)(4p-1)4p]$ to be non zero; for $3p$ dimensions, we can have $[123] = ... = [(3p-2)(3p-1)3p]$ to be non zero; where $p$ is an integer.

Examples 1-3 satisfy these conditions for $k = 4$.

Example 7. Consider a 5-dimensional design with the following points:

1. 80 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u}, x_{5u})$ with coordinates

   $$(\pm x, \pm y, \pm z, \pm w, \pm s)$$

   $$(\pm y, \pm z, \pm w, \pm s, \pm x)$$

   $$(\pm z, \pm w, \pm s, \pm x, \pm y)$$

   $$(\pm w, \pm s, \pm x, \pm y, \pm z)$$

   $$(\pm s, \pm x, \pm y, \pm z, \pm w)$$

   where only combinations of signs are used which make: $x_{1u}x_{2u}x_{3u}x_{4u}x_{5u} = x y z w s$.

2. 10 axial points with distance $\alpha$.

3. $n_0$ center points.
In this case, \([ii]N = 16(x^2 + y^2 + z^2 + w^2 + s^2) + 2 \alpha^2\), \([iii]N = 16 \left(x^4 + y^4 + z^4 + w^4 + s^4\right) + 2 \alpha^4\), \([iijj]N = 16 \left(z^2 w^2 + w^2 s^2 + x^2 y^2 + y^2 z^2 + s^2 x^2\right)\) for \((i, j) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\), and the other \([iijj]N = 16 \left(x^2 z^2 + y^2 w^2 + z^2 s^2 + w^2 x^2 + s^2 y^2\right)\). Other moments of orders up to and including 4 are zero. The design, with \((90 + n_0)\) points, is SROAD for any values of \(x, y, z, w, s, \alpha\), and any number of center points.

**Example 8.** Consider this 6-dimensional design:

1. 48 points \((x_{1u}, x_{2u}, x_{3u}, x_{4u}, x_{5u}, x_{6u})\) with coordinates

\[
(\pm x, \pm y, \pm z, \pm x, \pm y, \pm z)
\]
\[
(\pm y, \pm z, \pm x, \pm y, \pm z, \pm x)
\]
\[
(\pm z, \pm x, \pm y, \pm z, \pm x, \pm y)
\]

where only combinations of signs are used which make: \(x_{1u} x_{2u} x_{3u} = x_{4u} x_{5u} x_{6u} = x y z\).

2. 12 axial points with distance \(\alpha\).

3. \(n_0\) center points.

In this case, \([ii]N = 16 (x^2 + y^2 + z^2) + 2 \alpha^2\), \([123] = [456] = 48 x y z\), are non-zero odd moments, \([iii]N = 16 \left(x^4 + y^4 + z^4\right) + 2 \alpha^4\), \([iijj]N = 16 \left(x^4 + y^4 + z^4\right)\) for \((i, j) = (1, 4), (2, 5), (3, 6)\), and the other \([iijj]N = 16 \left(x^2 y^2 + y^2 z^2 + z^2 x^2\right)\). Other moments of orders up to and including 4 are zero. The design, which has \((60 + n_0)\) points, is SROAD for any values of \(x, y, z, \alpha\), and any number of center points.

As we have seen, there exists a much wider range of SROAD designs than just the symmetrical designs given by Park.
Acknowledgements

N. R. Draper gratefully acknowledges partial support from the Wisconsin Alumni Research Foundation via the Graduate School of the University of Wisconsin, from the National Security Agency via Grant No. MDA904-92-H-3096 and from the German Alexander von Humboldt Stiftung.

References


