General solution for the economic design of $\bar{X}$-charts based on Duncan's model

by

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Abstract

An exact and relatively simple general solution for the design of $X$-charts based on Duncan's model is derived in this paper. The general solution consists of an implicit equation in design variables $n$ (sample size) and $k$ (control limit factor) and an explicit equation for $h$ (sampling interval). The use of this general solution not only yields the exact optimum but also provides valuable information so that the sensitivity of the optimum loss-cost ($L^*$) can be evaluated. Loss-cost contours are used to discuss the nature of the loss-cost surface and the effect of the design variables. The effect of two parameters, the delay factor ($e$), and the average time for an assignable cause to occur ($1/\lambda$), on the optimum design is evaluated. Numerical examples are used for illustrations.
INTRODUCTION

Since the pioneering work of Shewhart (5), many authors have discussed the economic aspects of the design of control charts. Most of them (4, 6) have been concerned with the minimization of inspection, while some (1, 2, 3) have considered plans based on maximum income criterion.

Based on the maximum income criterion, Duncan (2) has formulated a model which measures the average net income of a process under the surveillance of an $\bar{X}$-chart. The procedure consists of taking samples of size $n$ at regular intervals of $h$ hours and measuring some characteristic $x$ of the articles, the sample mean $\bar{x}$ is calculated and recorded on an $\bar{X}$-chart. If this sample $\bar{x}$ falls outside the control limits, $\bar{X} + k \sigma / \sqrt{n}$, it is assumed that some shift in the process average has occurred and a search for an assignable cause is undertaken. The practical problem, then, is whether to take large samples at less frequent intervals or small samples at more frequent intervals. This decision depends not only on the various risks inherent in the sampling process but also on the cost of inspection and of scrap and rework. Thus, the problem is to find a decision policy which minimizes the long term costs.

Duncan's solution for the optimum values of these parameters is based on certain approximations and assumptions which lead to some limitations, and the scheme of finding the optimum solution is fairly complicated and involved. In this paper a general solution, based on

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Duncan's model, is derived for determining the optimum values of \( n, k, \) and \( h \). The general solution consists of an implicit equation in variables \( n \) and \( k \) and an explicit equation for \( h \) in terms of \( n \) and \( k \). Comparison is made between Duncan's approximate method and the new general solution. Finally the loss-cost surfaces are analyzed and the effects of two parameters, \( e \) and \( \lambda \), are evaluated.

2. **DUNCAN'S MODEL AND APPROXIMATE METHOD**

Duncan's model uses a simple economic principle that average net income = income - cost, where income is divided into two parts

(a) income when the process is in control, \( V_0 \)

(b) income when the process is out of control, \( V_1 \).

Similarly, cost incurred in the process is divided into three parts

(a) average cost of finding the assignable cause when none exists, \( T \)

(b) average cost of finding the assignable cause when it exists, \( W \)

(c) cost of maintaining the control chart, \( (b + cn) / h \).

In order to develop an expression for income, proportion of the time when the process is in control, \( \alpha \), and proportion of the time when it is out of control, \( \gamma \), have been derived in the original paper. For calculating the cost factor, average number of times the process actually goes out of control, \( \varepsilon \), and expected number of false alarms, \( \beta a / h \), have been determined.
The average net income per hour, therefore, is given by

\[ I = (\beta V_o + \gamma V_1) - \left( \frac{\alpha}{h} \right) T + \varepsilon W + \frac{(b + cn)}{h} \]  
...... (2.1)

which can be written as,

\[ I = V_o - L \]

where

\[ L = \frac{(1/P - 1/2 + \lambda h/12) \lambda hM + (en + D) \lambda M + \alpha T/h + \lambda W}{(1/P - 1/2 + \lambda h/12)h + (en + D) \lambda + 1} + \frac{(b + cn)}{h} \]  
...... (2.2)

The quantity \( L \) is called the loss-cost. The income, \( I \), will approximately be a maximum for \( n, k, \) and \( h \) when \( L \) is a minimum; therefore the criterion of an optimum design is the minimum loss-cost.

Based on a few assumptions and approximations, which will be discussed in later sections, Duncan derived the following approximate solutions.

\[ h = \sqrt{\frac{e^T + b + cn}{\lambda M(1/P - 1/2)}} \]  
...... (2.3)

\[ -n + \frac{\beta^2(1/P - 1/2)}{\alpha P/\alpha n} = (\alpha T + b)/c \]  
...... (2.4)

\[ \exp \left( -\frac{k^2}{2} \right) / \sqrt{2\pi} = c \sqrt{n}/\delta T \]  
...... (2.5)

Equations (2.4) and (2.5) are solved for \( n \) and \( k \) by using graphical method and equation (2.3) gives \( h \) explicitly in terms of \( n \) and \( k \). Optimum values of \( n, k, \) and \( h \) for a given case are, thus, obtained from equations (2.3), (2.4), and (2.5) by trial and error method.
The average net income per hour, therefore, is given by

\[ I = \left( \beta V_0 + \gamma V_1 \right) - \left( \frac{\alpha}{h} T + \varepsilon W + \left( b + cn \right) / h \right) \quad \ldots \ldots \quad (2.1) \]

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\[ L = \frac{(1/P - 1/2 + \lambda h/12) \lambda hM + (en + D)\lambda M + \alpha T/h + \lambda W}{(1/P - 1/2 + \lambda h/12) h + (en + D) \lambda + 1} + \left( b + cn \right) / h \quad \ldots \ldots \quad (2.2) \]

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\[ h = \sqrt{\frac{\alpha T + b + cn}{\lambda M(1/P - 1/2)}} \quad \ldots \ldots \quad (2.3) \]

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\[ \exp \left( - k^2/2 \right) / \sqrt{2\pi} = c \sqrt{n} / \delta T \quad \ldots \ldots \quad (2.5) \]

Equations (2.4) and (2.5) are solved for \( n \) and \( k \) by using graphical method and equation (2.3) gives \( h \) explicitly in terms of \( n \) and \( k \). Optimum values of \( n, k, \) and \( h \) for a given case are, thus, obtained from equations (2.3), (2.4), and (2.5) by trial and error method.
Duncan's model, is derived for determining the optimum values of $n$, $k$, and $h$. The general solution consists of an implicit equation in variables $n$ and $k$ and an explicit equation for $h$ in terms of $n$ and $k$. Comparison is made between Duncan's approximate method and the new general solution. Finally the loss-cost surfaces are analyzed and the effects of two parameters, $e$ and $\lambda$, are evaluated.

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(c) cost of maintaining the control chart, $(b + cn) / h$.

In order to develop an expression for income, proportion of the time when the process is in control, $\beta$, and proportion of the time when it is out of control, $\gamma$, have been derived in the original paper. For calculating the cost factor, average number of times the process actually goes out of control, $e$, and expected number of false alarms, $\beta a / h$, have been determined.
3. GENERAL SOLUTION

Based on Duncan's model a general solution is derived for finding the exact optimum values of sample size, (n), control limit factor, (k), and the sampling interval, (h). Mathematical derivation of the general solution is given below and its use is illustrated by a numerical example in the next section.

The loss-cost, as given by equation (2.2), can be written as:

\[
L = \frac{(xM + \alpha T/h + \lambda W)}{(1 + x)} + \frac{(b + cn)}{h} \quad \ldots \ldots \ldots \quad (3.1)
\]

where \( x = (1/P - 1/2 + \lambda h/12) \lambda h + (en + D)\lambda \) \( \ldots \ldots \ldots \quad (3.2) \)

For a given n, the following conditions should be satisfied for \( L \) to be a minimum.

\[
\frac{\partial L}{\partial k} = 0 \quad \ldots \ldots \quad (3.3)
\]

and \( \frac{\partial L}{\partial h} = 0 \quad \ldots \ldots \quad (3.4) \)

The above conditions yield equations (3.5) and (3.6) respectively as shown in Appendix A.

\[
\lambda = \frac{\exp\left(- \left(\frac{k - \delta n}{\sqrt{n}}\right)^2/2\right)}{2TP^2 \left\{ \exp\left(-k^2/2\right) \right\}} = \frac{(1 + x)}{(M - \lambda W)h^2 - \alpha Th} \quad \ldots \ldots \quad (3.5)
\]

\[
\lambda^2 h/6 + (1/P - 1/2) \lambda = \frac{(1 + x)}{(M - \lambda W)h^2 - \alpha Th} \left\{ \alpha T + (1 + x)(b + cn) \right\} \quad \ldots \ldots \quad (3.6)
\]
From equations (3.5) and (3.6) we obtain two quadratic equations as follows. The first one is obtained by substituting the values of 
\[(1 + x) / (M - \lambda W) h^2 - aTh\] from equation (3.5) and of \(x\) from equation (3.2) in equation (3.6). On simplification we get:

\[
\left[ \lambda^2 (b + cn)/12 \right] h^2 + \left[ (1/P - 1/2)(b + cn) - (2T_\lambda P^2 \exp(-k^2/2)) \right] h \\
+ \left[ (\lambda en + \lambda D + 1)(b + cn) - (1/P - 1/2)2TP^2 \exp(-k^2/2) \right] = 0
\]

\[\text{(3.7)}\]

Another quadratic equation is obtained by substituting the value of \(x\) from (3.2) in (3.5) and we have

\[
\left[ \frac{\lambda (M - \lambda W) \exp\{(k - \delta \sqrt{n})^2/2\}}{P^2} - \lambda^2 T \exp(-k^2/2) / 6 \right] h^2 \\
+ \left[ -\frac{\alpha \lambda T}{P^2} \exp\{(k - \delta \sqrt{n})^2/2\} - 2T_\lambda (1/P - 1/2) \exp(-k^2/2) \right] h \\
+ \left[ -2T(\lambda en + \lambda D + 1) \exp(-k^2/2) \right] = 0
\]

\[\text{(3.8)}\]

Equations (3.7) and (3.8) can be written as

\[
\beta_1 h^2 + \beta_2 h + \beta_3 = 0
\]

\[\text{(3.9)}\]

\[
\gamma_1 h^2 + \gamma_2 h + \gamma_3 = 0
\]

\[\text{(3.10)}\]

by letting

\[
\beta_1 = \lambda^2 (b + cn) / 12
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From equations (3.5) and (3.6) we obtain two quadratic equations as follows. The first one is obtained by substituting the values of

\[\frac{(1+x)}{((M-\lambda W)\hbar^2 - \alpha Th)}\] from equation (3.5) and of \(x\) from equation (3.2) in equation (3.6). On simplification we get:

\[
\begin{align*}
\left[\lambda^2(b+cn)/12\right]\hbar^2 + \left[(1/P-1/2)(b+cn)\lambda - \frac{(2T_\lambda P^2 \cdot \exp(-k^2/2)}{6\exp(-(k-\delta \sqrt{n})^2/2)}\right]h
+ \left[(\lambda en + \lambda D + 1)(b+cn) - (1/P-1/2)2TP^2 \frac{\exp(-k^2/2)}{\exp(-(k-\delta \sqrt{n})^2/2)} + \alpha T\right] = 0
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\begin{align*}
\left[\lambda (M-\lambda W) \exp(-(k-\delta \sqrt{n})^2/2)\right]P^2 - \lambda^2 T \exp(-k^2/2)/6\right] h^2
+ \left[-\frac{\alpha T}{\rho_2^2} \exp(-(k-\delta \sqrt{n})^2/2)\right] -2T \lambda (1/\rho_2 - 1/2) \exp(-k^2/2)\right] h
+ \left[-2T(\lambda en + \lambda D+1) \exp(-k^2/2)\right] = 0
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\]

\[
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\]

\[
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\]

\[
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The above conditions yield equations (3.5) and (3.6) respectively as shown in Appendix A.

\[
\frac{\lambda}{2TP^2} \cdot \exp \left\{ - \frac{(k - \delta \sqrt{n})^2}{2} \right\} = \frac{(1 + x)}{(M - \lambda W)^2 - \alpha Th} \quad \ldots \ldots (3.5)
\]

\[
\lambda ^2 h/6 + \frac{(1/P - 1/2) \lambda}{(M - \lambda W)^2 - \alpha Th} = \frac{(1 + x)}{(M - \lambda W)^2 - \alpha Th} \quad \ldots \ldots (3.6)
\]
\[ \beta_2 = \frac{(1/P - 1/2)(b + cn)\lambda - (2T\lambda P^2/6)}{\exp\left(-k^2/2\right)} + \frac{\exp\left(-k^2/2\right)}{\exp\left(-(k - \sqrt{\lambda})^2/2\right)} \]

\[ \beta_3 = (\lambda e\sqrt{P} + \lambda D + 1)(b + cn) - (1/P - 1/2)(2TP^2) + \frac{\exp(-k^2/2)}{\exp\left(-(k - \sqrt{\lambda})^2/2\right)} + \alpha T \]

\[ \gamma_1 = \frac{\lambda(M - \lambda W)}{p^2} \exp\left\{ -(k - \sqrt{\lambda})^2/2 \right\} - \lambda^2 T \exp(-k^2/2)/6 \]

\[ \gamma_2 = -\frac{\alpha \lambda T}{p^2} \exp\left\{ -(k - \sqrt{\lambda})^2/2 \right\} - 2T\lambda (1/P - 1/2) \exp(-k^2/2) \]

\[ \gamma_3 = -2T(\lambda e\sqrt{P} + \lambda D + 1) \exp(-k^2/2) \]

Since the coefficients \( \beta \)'s and \( \gamma \)'s in equations (3.9) and (3.10) are functions of \( n \) and \( k \) alone, we can eliminate \( h \) to obtain,

\[ (\gamma_1\beta_2 - \gamma_2\beta_1)(\gamma_2\beta_3 - \gamma_3\beta_2) - (\gamma_1\beta_3 - \gamma_3\beta_1)^2 = 0 \ldots \]

(3.11)

or \( f(n,k) = 0 \)

which is an implicit equation in variables \( n \) and \( k \). Also from equations (3.9) and (3.10) we obtain

\[ h = \frac{(\beta_3 \gamma_1 - \beta_1 \gamma_3)}{(\beta_1 \gamma_2 - \beta_2 \gamma_1)} \ldots \]

(3.12)

This gives \( h \) in terms of \( n \) and \( k \). The basic equations comprising the general solution are (3.11) and (3.12)

4. NUMERICAL METHOD

Equation (3.11) cannot be solved explicitly for either \( n \) or \( k \) because \( f(n,k) \) is a complicated function and the probability integrals
P and a cannot be evaluated without knowing \( n \) and \( k \) both. However, it is possible to find one or more values of \( k \), by numerical method, which will satisfy (3.11) to any desired degree of accuracy for known values of the cost and risk factors and for an assumed value of \( n \). Assuming that the values of \( k \) are within the range between 0.1 and 5.0 for practical purposes, we employ a numerical procedure for obtaining the optimum values \( n^*, k^* \) and \( h^* \) of the design variables \( n \), \( k \) and \( h \) respectively as follows:

(i) Assume an initial integer \( n \), say \( n_0 \) \((n_0 \geq 1)\)

(ii) For \( n = n_0 \), obtain values of \( k \) that satisfy equation (3.11) as closely as desired.

(iii) Calculate \( h \) for each of these \( k \) values from equation (3.12)

(iv) Using only the real, positive, \( h \) values from (iii) (say \( h_1, h_2, h_3, \ldots \)) and the associated \( k \) values (say \( k_1, k_2, k_3, \ldots \)) obtain the local minimum loss-cost \( L^*_n \) from equation (3.1) where

\[
L^*_n = \min L(n_0, k_j, h_j), \quad j = 1, 2, 3, \ldots
\]

(v) Repeat (i) through (iv) for other values of \( n \) (say \( n_1, n_2, \ldots \)) and find the overall minimum loss-cost \( L^* \), which is given by

\[
L^* = L(n^*, k^*, h^*) = \min L^*_n, \quad r = 0, 1, 2, \ldots
\]

The values \( n^*, k^*, \) and \( h^* \) which give the overall minimum loss-cost, \( L^* \), are the optimum design values for the control chart.

As an illustration we find the optimum design for Example 1 which has the cost and risk factors as follows:
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(i) Assume an initial integer $n$, say $n_0$ ($n_0 \geq 1$)

(ii) For $n = n_0$, obtain values of $k$ that satisfy equation (3.11) as closely as desired.

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(iv) Using only the real, positive, $h$ values from (iii) (say $h_1, h_2, h_3, \ldots$) and the associated $k$ values (say $k_1, k_2, k_3, \ldots$) obtain the local minimum loss-cost $L^*_{n_0}$ from equation (3.1) where

$$L^*_{n_0} = \min_{n_0} L(n_0, k_j, h_j), \quad j = 1, 2, 3, \ldots$$

(v) Repeat (i) through (iv) for other values of $n$ (say $n_1, n_2, \ldots$) and find the overall minimum loss-cost $L^*$, which is given by

$$L^* = L(n^*, k^*, h^*) = \min_{n_r} L^*_{n_r}, \quad r = 0, 1, 2, \ldots$$

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As an illustration we find the optimum design for Example 1 which has the cost and risk factors as follows:
\[ \beta_2 = \frac{(1/p - 1/2)(b + cn) \lambda}{(2T \lambda P^2/6)} \frac{\exp(-k^2/2)}{\exp(-(k - \delta \sqrt{n})^2/2)} \]

\[ \beta_3 = (\lambda \ en + \lambda \ D + 1)(b + cn) - (1/P - 1/2)(2TP^2) \frac{\exp(-k^2/2)}{\exp(-(k - \delta \sqrt{n})^2/2)} + aT \]

\[ \gamma_1 = \frac{\lambda(M - \lambda W)}{p^2} \exp\{-(k - \delta \sqrt{n})^2/2\} - \lambda \ 2T \exp(-k^2/2)/6 \]

\[ \gamma_2 = -\frac{a \lambda T}{p^2} \exp\{-(k - \delta \sqrt{n})^2/2\} - 2T \lambda (1/P - 1/2) \exp(-k^2/2) \]

\[ \gamma_3 = -2T(\lambda \ en + \lambda \ D + 1) \exp(-k^2/2) \]

Since the coefficients \( \beta \)'s and \( \gamma \)'s in equations (3.9) and (3.10) are functions of \( n \) and \( k \) alone, we can eliminate \( h \) to obtain,

\[ (\gamma_1 \beta_2 - \gamma_2 \beta_1) (\gamma_2 \beta_3 - \gamma_3 \beta_2) - (\gamma_1 \beta_3 - \gamma_3 \beta_1)^2 = 0 \quad \ldots \quad (3.11) \]

or \( f(n, k) = 0 \)

which is an implicit equation in variables \( n \) and \( k \). Also from equations (3.9) and (3.10) we obtain

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\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$n_r$ & $k$ & $f(n,k)$ & $h$ & $L^*_{n_r}$ \\
\hline
1 & 2.5261 & $1.05 \times 10^{-9}$ & 0.6720 & 576.6843 \\
2 & 2.6911 & $1.46 \times 10^{-10}$ & 0.9246 & 458.1015 \\
3 & 2.8234 & $5.06 \times 10^{-10}$ & 1.1359 & 417.7678 \\
4 & 2.9524 & $3.07 \times 10^{-10}$ & 1.2881 & 404.0681 \\
5 & 3.0820 & $1.23 \times 10^{-11}$ & 1.4077 & 401.3788 \\
6 & 3.2112 & $6.96 \times 10^{-11}$ & 1.5001 & 404.2958 \\
7 & 3.3403 & $8.94 \times 10^{-10}$ & 1.5649 & 411.2918 \\
8 & 3.4653 & $2.65 \times 10^{-10}$ & 1.6556 & 419.6623 \\
9 & 3.5894 & $1.18 \times 10^{-9}$ & 1.7228 & 428.4784 \\
10 & 3.7113 & $4.28 \times 10^{-10}$ & 1.7859 & 437.8983 \\
\hline
\end{tabular}
\caption{Results for Example 1}
\end{table}
\[ \begin{align*}
\delta &= 2.00 & M &= $100 \\
\lambda &= 0.01 & T &= $50 \\
c &= 0.05 & W &= $25 \\
D &= 2.00 & b &= $.50 \\
c &= $.10
\end{align*} \]

We assume an initial value of \( n \) equal to 1 and find all values of \( k \) between 0.1 and 5.0 that satisfy equation (3.11) to \( 10^{-8} \) or better. For each \( k \), the value of \( h \) is calculated from equation (3.12) and the loss-cost, \( L^*_1 \) is computed as described above. Thus, we obtain

\[ f(n,k) = 1.05 \times 10^{-9}, \quad k = 2.5262, \quad h = 0.6720 \quad \text{and} \quad L^*_1 = 576.68 \]

as the optimum design for \( n = 1 \). Following a similar procedure, we obtain designs for values of \( n \) from 2 to 10 as shown in Table 1. The overall optimum design is \( n^* = 5, \quad k^* = 3.08, \quad h^* = 1.41, \quad \text{and} \quad L^* = 401.38 \).

5. **COMPARISON OF GENERAL SOLUTION AND DUNCAN'S APPROXIMATE METHOD**

The advantages of the general solution can be summarized in three aspects as follows.

(a) **Accuracy:** In arriving at an approximate solution for the optimum values of \( n, k, \) and \( h \), Duncan makes a few assumptions such as

(i) terms like \( a T/h \) and \( \lambda [((1/P - 1/2 + \lambda h/12)h + en + D] \) are neglected.

(ii) \( \lambda \) is assumed small and all terms in an equation of a smaller order of magnitude than the principal term are also neglected.
\[
\begin{align*}
\delta &= 2.00 & M &= \$100 \\
\lambda &= 0.01 & T &= \$50 \\
c &= 0.05 & W &= \$25 \\
D &= 2.00 & b &= \$0.50 \\
c &= \$0.10
\end{align*}
\]

We assume an initial value of \( n \) equal to 1 and find all values of \( k \) between 0.1 and 5.0 that satisfy equation (3.11) to \( 10^{-8} \) or better. For each \( k \), the value of \( h \) is calculated from equation (3.12) and the loss-cost, \( L^* \) is computed as described above. Thus, we obtain

\[ f(n,k) = 1.05 (10)^{-9}, \quad k = 2.5262, \quad h = 0.6720 \text{ and } L^* = 576.68 \]

as the optimum design for \( n = 1 \). Following a similar procedure, we obtain designs for values of \( n \) from 2 to 10 as shown in Table 1. The overall optimum design is \( n^* = 5, \quad k^* = 3.08, \quad h^* = 1.41 \), and \( L^* = 401.38 \).

5. COMPARISON OF GENERAL SOLUTION AND DUNCAN'S APPROXIMATE METHOD

The advantages of the general solution can be summarized in three aspects as follows.

(a) **Accuracy:** In arriving at an approximate solution for the optimum values of \( n, k, \) and \( h \), Duncan makes a few assumptions such as

(i) terms like \( \alpha T/h \) and \( \lambda \left[(1/P - 1/2 + \lambda h/12)h + en + D\right] \) are neglected.

(ii) \( \lambda \) is assumed small and all terms in an equation of a smaller order of magnitude than the principal term are also neglected.


**TABLE 1**  
Results for Example 1

<table>
<thead>
<tr>
<th>$n_r$</th>
<th>$k$</th>
<th>$f(n,k)$</th>
<th>$h$</th>
<th>$I_{n_r}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5261</td>
<td>1.05e-9</td>
<td>0.6720</td>
<td>576.6843</td>
</tr>
<tr>
<td>2</td>
<td>2.6911</td>
<td>1.46e-10</td>
<td>0.9246</td>
<td>458.1015</td>
</tr>
<tr>
<td>3</td>
<td>2.8234</td>
<td>5.06e-10</td>
<td>1.1359</td>
<td>417.7678</td>
</tr>
<tr>
<td>4</td>
<td>2.9524</td>
<td>3.07e-10</td>
<td>1.2881</td>
<td>404.0681</td>
</tr>
<tr>
<td>5</td>
<td>3.0820</td>
<td>1.23e-11</td>
<td>1.4077</td>
<td>401.3788</td>
</tr>
<tr>
<td>6</td>
<td>3.2112</td>
<td>6.96e-11</td>
<td>1.5001</td>
<td>404.8958</td>
</tr>
<tr>
<td>7</td>
<td>3.3403</td>
<td>8.94e-10</td>
<td>1.5649</td>
<td>411.2918</td>
</tr>
<tr>
<td>8</td>
<td>3.4653</td>
<td>2.65e-10</td>
<td>1.6556</td>
<td>419.4623</td>
</tr>
<tr>
<td>9</td>
<td>3.5894</td>
<td>1.18e-9</td>
<td>1.7228</td>
<td>428.4784</td>
</tr>
<tr>
<td>10</td>
<td>3.7113</td>
<td>4.28e-10</td>
<td>1.7859</td>
<td>437.8982</td>
</tr>
</tbody>
</table>
(iii) no account is taken of the values of the cost and risk factors $e$, $D$, and $W$.

The general solution, however, is free from these assumptions and approximations. No terms are neglected for finding the solution. Therefore it gives accurate and reliable optimum values for the design parameters.

(b) **Applicability**

As pointed out by Duncan, some of the limitations of the approximate method are:

(i) For relatively high values of $e$ it does not seem to work as well as for low values.

(ii) For small values of $\delta$ (e.g. $\delta = 0.5$) approximate optima can be determined only very roughly due to the difficulty in interpolation and extrapolation of the curves by the graphical method.

These limitations on the values of $e$ and $\delta$ restrict the applicability of the approximate method and, as discussed below, lead to significantly different designs from the true optima in some cases. In the general solution no such limitations are imposed on the values of the cost and risk factors or other parameters. No interpolation or extrapolation is involved and, hence, no difficulty is encountered in finding an optimum design for high values of $e$ or low values of $\delta$.

(c) **Flexibility**

An important characteristic of the general solution is that it gives not only the exact optimum design but also furnishes additional information
about its neighborhood. This provides a means of evaluating the sensi-
tivity of the optimum loss-cost to changes in the design variables and,
hence, gives a flexibility in the choice of these variables for a known addi-
tional loss-cost. A discussion of this feature along with numerical examples
is given in the next section.

Optimum designs for 15 representative examples from Duncan's paper
were obtained by using the general solution. Also loss-cost figures for
these examples were recalculated for Duncan's values of n, k, and h.
Based on these results (Table 2) we note that:

(i) In all the cases considered, the general solution yields lower loss-
costs than the approximate method.

(ii) For high values of loss rate, M, the design values by the general
solution yield substantially lower loss-costs. For instance in Example
22, the approximate optimum loss-cost value is 12% higher than the
true optimum. Also there is a difference of $60 per 100 hours in
the loss-cost for Example 5 for which the value of M is $1000.
(Appendix B)

(iii) As pointed out above, the approximate optimum for a high value of
e yields higher loss-cost as revealed by Example 7 (e = 0.50). The
true optimum loss-cost for this example is approximately 11% lower
than the approximate optimum.

(iv) The difference between the results becomes more pronounced for
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An important characteristic of the general solution is that it gives not only the exact optimum design but also furnishes additional information
### TABLE 2

Comparison of Results by General Solution and Duncan's Methods

<table>
<thead>
<tr>
<th>Group</th>
<th>Example No.</th>
<th>General Solution</th>
<th>Duncan's Approximate Method</th>
<th>Duncan's Exact Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n</td>
<td>k</td>
<td>h</td>
<td>L</td>
</tr>
<tr>
<td>I</td>
<td>1</td>
<td>5</td>
<td>3.08</td>
<td>1.41</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>3.08</td>
<td>1.02</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>2.94</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>2.95</td>
<td>0.41</td>
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<td>7</td>
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<td>3.05</td>
<td>1.62</td>
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<td>10</td>
<td>6</td>
<td>3.67</td>
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<td>12</td>
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<td>2.88</td>
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<td>14</td>
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<td>1.46</td>
<td>4.66</td>
</tr>
<tr>
<td>II</td>
<td>16</td>
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<td>5.47</td>
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<td>3.39</td>
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<tr>
<td></td>
<td>19</td>
<td>18</td>
<td>2.56</td>
<td>11.02</td>
</tr>
<tr>
<td>III</td>
<td>21</td>
<td>38</td>
<td>2.21</td>
<td>23.45</td>
</tr>
<tr>
<td></td>
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<td>2.11</td>
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<td></td>
<td>25</td>
<td>12</td>
<td>1.13</td>
<td>54.32</td>
</tr>
</tbody>
</table>

Note: (1) L' are the recalculated loss-cost values for Duncan's design parameters n, k, and h
(2) Values not available
(3) All loss-cost values are for 100 hours of operation
low values of $\delta$. For instance, the optimum designs by general solution for Examples 21, 22, and 25 ($\delta = 0.5$) are considerably different from the approximate optima. The approximate loss-cost values for Examples 22 and 25 are higher than the true optima by 12% and 6% respectively.

(v) The recalculated loss-cost values for Duncan's design parameters are somewhat different from his reported values. In some cases (Examples 5, 10, 18, 21 and 25) the recalculated values are lower while in others they are higher. There is also a slight difference for his exact method loss-cost figures.

6. ANALYSIS OF LOSS-COST SURFACES

As mentioned earlier, the general solution provides the optimum designs and useful additional information which enables us to evaluate the sensitivity of the optimum loss-cost in terms of $n$, $k$, and $h$. Example 1 is used to illustrate the flexibility of the choice of sample sizes and for an evaluation of the loss-cost surfaces.

Recall that the optimum sample size for Example 1 is 5 at the optimum loss-cost of $401.38 as shown in Table 1 together with the complete $L_n^r$'s for different sample sizes. Note that a deviation of one unit from the optimum sample size causes an increase of the loss-cost by less than 1%. This means that any sample size between 4 to 6 along with the appropriate $k$ and $h$ values (shown in Table 1) can be used by incurring an additional loss-cost within 1% of the minimum value. If the sample size varies
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<th>Example No.</th>
<th>General Solution</th>
<th>Duncan's Approximate Method</th>
<th>Duncan's Exact Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>n    k    h    L</td>
<td>n    k    h    L</td>
<td>L'(1)</td>
</tr>
<tr>
<td>I (δ=2.0)</td>
<td>1</td>
<td>5    3.08 1.41 401.38</td>
<td>5 3.2 1.3 402.13</td>
<td>402.20</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5 3.08 1.02 694.77</td>
<td>5 3.2 0.9 695.55</td>
<td>696.64</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4 2.94 0.78 959.47</td>
<td>5 3.2 0.3 962.39</td>
<td>962.44</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4 2.95 0.41 2697.63</td>
<td>5 3.2 0.3 2756.67</td>
<td>2756.42</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>2 2.69 0.94 541.16</td>
<td>5 3.2 1.3 608.86</td>
<td>608.93</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>5 3.05 1.62 1837.28</td>
<td>5 3.2 1.3 1840.78</td>
<td>1840.83</td>
</tr>
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<td>6 3.67 1.45 637.05</td>
<td>6 3.8 1.3 659.37</td>
<td>638.60</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>6 2.88 3.47 586.95</td>
<td>7 3.2 3.3 589.86</td>
<td>589.89</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>1 1.46 4.66 990.99</td>
<td>2 1.4 7.0 1010.74</td>
<td>1011.58</td>
</tr>
<tr>
<td>II (δ=1.0)</td>
<td>16</td>
<td>14 2.68 5.47 141.80</td>
<td>17 2.8 5.6 142.82</td>
<td>142.85</td>
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<tr>
<td></td>
<td>18</td>
<td>21 3.39 7.23 364.29</td>
<td>22 3.5 6.0 365.83</td>
<td>365.79</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>18 2.56 11.02 195.78</td>
<td>22 2.7 10.6 197.43</td>
<td>197.43</td>
</tr>
<tr>
<td>III (δ=0.5)</td>
<td>21</td>
<td>38 2.21 23.45 83.70</td>
<td>46 2.3 22.0 94.97</td>
<td>84.52</td>
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<tr>
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<td>22</td>
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<td>1526.12</td>
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<tr>
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<td>12 1.13 54.32 132.65</td>
<td>20 1.3 44.0 141.60</td>
<td>140.73</td>
</tr>
</tbody>
</table>

Note: (1) L' are the recalculated loss-cost values for Duncan's design parameters n, k, and h
(2) Values not available
(3) All loss-cost values are for 100 hours of operation
Figure 1: Optimum Values Of L, k, and h For Varying Sample Size (Example 1)
Figure 1: Optimum Values Of $L$, $k$, and $h$ For Varying Sample Size (Example 1)
Figure 2: Loss-Cost Contours For n = 4, 5, and 6 (Example 1)
<table>
<thead>
<tr>
<th>Example No.</th>
<th>( \lambda )</th>
<th>( e )</th>
<th>Optimum Design by General Solution</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.01</td>
<td>.05</td>
<td>( n^* ) 3.08 ( k^* ) 1.41</td>
<td>401.38</td>
</tr>
<tr>
<td>1a</td>
<td>&quot;</td>
<td>.10</td>
<td>4 2.95 1.29</td>
<td>422.79</td>
</tr>
<tr>
<td>1b</td>
<td>&quot;</td>
<td>.15</td>
<td>4 2.95 1.29</td>
<td>441.43</td>
</tr>
<tr>
<td>1c</td>
<td>&quot;</td>
<td>.30</td>
<td>3 2.82 1.14</td>
<td>487.46</td>
</tr>
<tr>
<td>7</td>
<td>&quot;</td>
<td>.50</td>
<td>2 2.69 0.94</td>
<td>541.16</td>
</tr>
<tr>
<td>1</td>
<td>.01</td>
<td>.05</td>
<td>5 3.08 1.41</td>
<td>401.38</td>
</tr>
<tr>
<td>2</td>
<td>.02</td>
<td>&quot;</td>
<td>5 3.08 1.02</td>
<td>694.77</td>
</tr>
<tr>
<td>3</td>
<td>.03</td>
<td>&quot;</td>
<td>4 2.94 0.78</td>
<td>959.47</td>
</tr>
</tbody>
</table>

**TABLE 3**

Optimum Designs for Different Values of \( e \) and \( \lambda \)

- only Parameter \( e \) changes
- only Parameter \( \lambda \) changes
<table>
<thead>
<tr>
<th>Example No.</th>
<th>( \lambda )</th>
<th>( e )</th>
<th>Optimum Design by General Solution</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.01</td>
<td>.05</td>
<td>5 ( n^* ) 3.08 ( k^* ) 1.41 ( h^* ) 401.38</td>
<td></td>
</tr>
<tr>
<td>1a</td>
<td>&quot;</td>
<td>.10</td>
<td>4 ( n^* ) 2.95 ( k^* ) 1.29 ( h^* ) 422.79</td>
<td>only Parameter ( e ) changes</td>
</tr>
<tr>
<td>1b</td>
<td>&quot;</td>
<td>.15</td>
<td>4 ( n^* ) 2.95 ( k^* ) 1.29 ( h^* ) 441.43</td>
<td></td>
</tr>
<tr>
<td>1c</td>
<td>&quot;</td>
<td>.30</td>
<td>3 ( n^* ) 2.82 ( k^* ) 1.14 ( h^* ) 487.46</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>&quot;</td>
<td>.50</td>
<td>2 ( n^* ) 2.69 ( k^* ) 0.94 ( h^* ) 541.16</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.01</td>
<td>.05</td>
<td>5 ( n^* ) 3.08 ( k^* ) 1.41 ( h^* ) 401.38</td>
<td>only Parameter ( \lambda ) changes</td>
</tr>
<tr>
<td>2</td>
<td>.02</td>
<td>&quot;</td>
<td>5 ( n^* ) 3.08 ( k^* ) 1.02 ( h^* ) 694.77</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.03</td>
<td>&quot;</td>
<td>4 ( n^* ) 2.94 ( k^* ) 0.78 ( h^* ) 959.47</td>
<td></td>
</tr>
</tbody>
</table>
the sample size. To examine its effect on the optimum design, we consider Examples 1, 1a, 1b, 1c and 7, for which the values on the delay factor are .05, .10, .15, .30, and .50 respectively, other cost and risk factors being the same. The optimum designs for these Examples (Table 3) show that a change in the value of e affects all three design variables \( n, k, \) and \( h \). When \( e \) is increased from 0.05 to 0.50, i.e. from 3 minutes to 30 minutes, the optimum sample size, \( n^* \), changes from 5 to 2, \( k^* \), from 3.08 to 2.69, and \( h^* \), drops from 1.41 to 0.94 hours. The optimum loss-cost increases from $401.38 to $541.16. The optimum values of the design parameters decrease and the optimum loss-cost increases as \( e \) increases. Therefore, as the value of the delay factor increases, the optimum design requires taking smaller samples at less frequent intervals and using tighter control limits.

To evaluate the effect of \( e \) on the loss-cost, we calculate a quantity called "Percentage Increase in Loss-Cost, PIL", which we define as follows:

\[
PIL, \text{ for a sample size } n_r, = \frac{L_{n_r}^* - L^*}{L^*} \times 100
\]

For instance in Example 1 we have \( L_4^* = \$404.07 \), and \( L^* = \$401.38 \), Therefore PIL, for a sample size 4, = \( \frac{404.07 - 401.38}{401.38} \times 100 = 0.67 \)

Computations for other sample sizes and other data points are made in a similar way and are summarized in Table 4.
<table>
<thead>
<tr>
<th>( n_r )</th>
<th>Example 1</th>
<th>Example 1a</th>
<th>Example 1b</th>
<th>Example 1c</th>
<th>Example 7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( L^*_{n_r} )</td>
<td>( PIL )</td>
<td>( L^*_{n_r} )</td>
<td>( PIL )</td>
<td>( L^*_{n_r} )</td>
</tr>
<tr>
<td>1</td>
<td>576.68</td>
<td>43.67</td>
<td>581.26</td>
<td>37.48</td>
<td>585.83</td>
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<tr>
<td>2</td>
<td>458.10</td>
<td>14.13</td>
<td>467.40</td>
<td>10.55</td>
<td>476.68</td>
</tr>
<tr>
<td>3</td>
<td>417.77</td>
<td>4.08</td>
<td>431.79</td>
<td>2.13</td>
<td>445.77</td>
</tr>
<tr>
<td>4</td>
<td>404.07</td>
<td>0.67</td>
<td>422.79</td>
<td>0.00</td>
<td>441.43</td>
</tr>
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<td>0.47</td>
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<td>432.94</td>
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<td>460.82</td>
</tr>
<tr>
<td>7</td>
<td>411.29</td>
<td>2.47</td>
<td>443.97</td>
<td>5.01</td>
<td>476.49</td>
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<tr>
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<td>456.74</td>
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<td>493.72</td>
</tr>
<tr>
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<td>428.48</td>
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<td>470.33</td>
<td>11.24</td>
<td>511.82</td>
</tr>
<tr>
<td>10</td>
<td>437.90</td>
<td>9.10</td>
<td>484.32</td>
<td>14.55</td>
<td>530.28</td>
</tr>
</tbody>
</table>

*Note: Underlined values denote the overall optimum, \( L^* \).
<table>
<thead>
<tr>
<th>( n_r )</th>
<th>Example 1 ( L^* n_r )</th>
<th>PIL</th>
<th>Example 1a ( L^* n_r )</th>
<th>PIL</th>
<th>Example 1b ( L^* n_r )</th>
<th>PIL</th>
<th>Example 1c ( L^* n_r )</th>
<th>PIL</th>
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<td>0.98</td>
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<td>0.67</td>
<td>422.79</td>
<td>0.00</td>
<td>441.43</td>
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<td>0.88</td>
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<td>443.97</td>
<td>5.01</td>
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<td>7.94</td>
<td>572.53</td>
<td>17.45</td>
<td>697.55</td>
<td>28.90</td>
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<td>744.80</td>
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<td>470.33</td>
<td>11.24</td>
<td>511.82</td>
<td>15.55</td>
<td>634.16</td>
<td>30.09</td>
<td>792.44</td>
<td>46.43</td>
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<tr>
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<td>437.90</td>
<td>9.10</td>
<td>484.32</td>
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<td>530.28</td>
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<td>665.59</td>
<td>36.54</td>
<td>840.07</td>
<td>55.24</td>
</tr>
</tbody>
</table>

²Note: Underlined values denote the overall optimum, \( L^* \)
the sample size. To examine its effect on the optimum design, we consider Examples 1, 1a, 1b, 1c and 7, for which the values on the delay factor are .05, .10, .15, .30, and .50 respectively, other cost and risk factors being the same. The optimum designs for these Examples (Table 3) show that a change in the value of $e$ affects all three design variables $n$, $k$, and $h$. When $e$ is increased from 0.05 to 0.50, i.e. from 3 minutes to 30 minutes, the optimum sample size, $n^*$, changes from 5 to 2, $k^*$, from 3.08 to 2.69, and $h^*$, drops from 1.41 to 0.94 hours. The optimum loss-cost increases from $401.38 to $541.16. The optimum values of the design parameters decrease and the optimum loss-cost increases as $e$ increases. Therefore, as the value of the delay factor increases, the optimum design requires taking smaller samples at less frequent intervals and using tighter control limits.

To evaluate the effect of $e$ on the loss-cost, we calculate a quantity called "Percentage Increase in Loss-Cost, PIL", which we define as follows:

$$\text{PIL, for a sample size } n_r = \frac{L_{n_r}^* - L^*}{L^*} \times 100$$

For instance in Example 1 we have $L_4^* = 404.07$, and $L^* = 401.38$. Therefore PIL, for a sample size 4, = $\frac{404.07 - 401.38}{401.38} \times 100 = 0.67$

Computations for other sample sizes and other data points are made in a similar way and are summarized in Table 4.
Table:

<table>
<thead>
<tr>
<th>Example</th>
<th>Symbol</th>
<th>e</th>
<th>n*</th>
<th>L*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>O</td>
<td>.05</td>
<td>5</td>
<td>401.38</td>
</tr>
<tr>
<td>la</td>
<td>△</td>
<td>.10</td>
<td>4</td>
<td>422.79</td>
</tr>
<tr>
<td>lb</td>
<td>⊖</td>
<td>.15</td>
<td>4</td>
<td>441.43</td>
</tr>
<tr>
<td>lc</td>
<td>⊕</td>
<td>.30</td>
<td>3</td>
<td>487.46</td>
</tr>
<tr>
<td>7</td>
<td>□</td>
<td>.50</td>
<td>2</td>
<td>541.16</td>
</tr>
</tbody>
</table>

Figure 3: Effect Of e On Loss-Cost
On comparison we see that the PIL versus $n$ curves (Figure 3) gradually become steeper as the delay factor is changed from 0.05 to 0.50. This increase in slope implies that the higher the value of $e$, the larger the loss-cost incurred for a given deviation from the optimum sample size. To illustrate, we calculate the PIL when $n$ is increased, say, by 3 units from its optimum value for each of the five data points, as follows.

<table>
<thead>
<tr>
<th>Example</th>
<th>Delay Factor, $e$</th>
<th>$n^*$</th>
<th>$n^* + 3$</th>
<th>PIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>5</td>
<td>8</td>
<td>4.50</td>
</tr>
<tr>
<td>1a</td>
<td>0.10</td>
<td>4</td>
<td>7</td>
<td>5.01</td>
</tr>
<tr>
<td>1b</td>
<td>0.15</td>
<td>4</td>
<td>7</td>
<td>7.94</td>
</tr>
<tr>
<td>1c</td>
<td>0.30</td>
<td>3</td>
<td>6</td>
<td>11.49</td>
</tr>
<tr>
<td>7</td>
<td>0.50</td>
<td>2</td>
<td>5</td>
<td>12.38</td>
</tr>
</tbody>
</table>

Thus, for a change in $n$ from 5 to 8 for $e = 0.05$, the increase in $L$ is only 4.5% while a similar change for $e = 0.50$ from 2 to 5, causes an increase of 12.38%. For values of $e$ in between, there is a gradual increase in the PIL values as shown above.

Effect of $\lambda$

The average number of assignable causes per hour is denoted by $\lambda$ and an increase in $\lambda$ is equivalent to a decrease in the average time for an assignable cause to occur. To study the effect of this increase
On comparison we see that the PIL versus \( n \) curves (Figure 3) gradually become steeper as the delay factor is changed from 0.05 to 0.50. This increase in slope implies that the higher the value of \( e \), the larger the loss-cost incurred for a given deviation from the optimum sample size. To illustrate, we calculate the PIL when \( n \) is increased, say, by 3 units from its optimum value for each of the five data points, as follows.

<table>
<thead>
<tr>
<th>Example</th>
<th>Delay Factor, ( e )</th>
<th>( n^* )</th>
<th>( n^* + 3 )</th>
<th>PIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>5</td>
<td>8</td>
<td>4.50</td>
</tr>
<tr>
<td>1a</td>
<td>0.10</td>
<td>4</td>
<td>7</td>
<td>5.01</td>
</tr>
<tr>
<td>1b</td>
<td>0.15</td>
<td>4</td>
<td>7</td>
<td>7.94</td>
</tr>
<tr>
<td>1c</td>
<td>0.30</td>
<td>3</td>
<td>6</td>
<td>11.49</td>
</tr>
<tr>
<td>7</td>
<td>0.50</td>
<td>2</td>
<td>5</td>
<td>12.38</td>
</tr>
</tbody>
</table>

Thus, for a change in \( n \) from 5 to 8 for \( e = 0.05 \), the increase in \( L \) is only 4.5% while a similar change for \( e = 0.50 \) from 2 to 5, causes an increase of 12.38%. For values of \( e \) in between, there is a gradual increase in the PIL values as shown above.

**Effect of \( \lambda \)**

The average number of assignable causes per hour is denoted by \( \lambda \) and an increase in \( \lambda \) is equivalent to a decrease in the average time for an assignable cause to occur. To study the effect of this increase
Figure 3: Effect Of $e$ On Loss-Cost
### TABLE 5

Calculation of PIL for Evaluation of Effect of $1/\lambda$

<table>
<thead>
<tr>
<th>n</th>
<th>Example 1 $L_n^*$ PIL</th>
<th>Example 2 $L_n^*$ PIL</th>
<th>Example 3 $L_n^*$ PIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>576.68 43.67</td>
<td>923.33 32.90</td>
<td>1222.47 27.41</td>
</tr>
<tr>
<td>2</td>
<td>458.10 14.13</td>
<td>764.48 10.03</td>
<td>1036.92 8.07</td>
</tr>
<tr>
<td>3</td>
<td>417.77 4.08</td>
<td>711.98 2.48</td>
<td>976.95 1.82</td>
</tr>
<tr>
<td>4</td>
<td>404.07 0.67</td>
<td>695.85 0.16</td>
<td>959.47 0.00</td>
</tr>
<tr>
<td>5</td>
<td>401.38^A 0.00</td>
<td>694.77 0.00</td>
<td>961.06 0.17</td>
</tr>
<tr>
<td>6</td>
<td>404.90 0.89</td>
<td>702.15 1.06</td>
<td>972.06 1.31</td>
</tr>
<tr>
<td>7</td>
<td>411.29 2.47</td>
<td>713.58 2.71</td>
<td>987.66 2.94</td>
</tr>
<tr>
<td>8</td>
<td>419.46 4.50</td>
<td>727.16 4.66</td>
<td>1006.12 4.86</td>
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<tr>
<td>9</td>
<td>428.48 6.75</td>
<td>742.00 6.80</td>
<td>1025.87 6.92</td>
</tr>
<tr>
<td>10</td>
<td>437.90 9.10</td>
<td>757.40 9.01</td>
<td>1046.22 9.04</td>
</tr>
</tbody>
</table>

^A Note: Underlined values denote the overall optimum, $L_n^*$
Figure 4: Effect Of \( \lambda \) On Loss-Cost
Figure 4: Effect of $\lambda$ on Loss-Cost

<table>
<thead>
<tr>
<th>Example</th>
<th>Symbol</th>
<th>$\lambda$</th>
<th>$n^*$</th>
<th>$L^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>O</td>
<td>0.01</td>
<td>5</td>
<td>401.38</td>
</tr>
<tr>
<td>2</td>
<td>$\Delta$</td>
<td>0.02</td>
<td>5</td>
<td>694.77</td>
</tr>
<tr>
<td>3</td>
<td>$+$</td>
<td>0.03</td>
<td>4</td>
<td>959.47</td>
</tr>
</tbody>
</table>
### TABLE 5

Calculation of PIL for Evaluation of Effect of $1/\lambda$

<table>
<thead>
<tr>
<th>n</th>
<th>Example 1 L*&lt;sub&gt;n&lt;/sub&gt;</th>
<th>Example 1 PIL</th>
<th>Example 2 L*&lt;sub&gt;n&lt;/sub&gt;</th>
<th>Example 2 PIL</th>
<th>Example 3 L*&lt;sub&gt;n&lt;/sub&gt;</th>
<th>Example 3 PIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>576.68</td>
<td>43.67</td>
<td>923.33</td>
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<td>1222.47</td>
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<td>10.03</td>
<td>1036.92</td>
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<td>2.71</td>
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<td>757.40</td>
<td>9.01</td>
<td>1046.22</td>
<td>9.04</td>
</tr>
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</table>

\(^\Delta\) Note: Underlined values denote the overall optimum, L*
in \( \lambda \) on the optimum design, we consider Examples 1, 2, and 3, with values of \( \lambda = 0.01, 0.02, \) and \( 0.03 \) respectively. If \( \lambda \) increases from 0.01 to 0.02, i.e. if the average time for an assignable cause to occur reduces from 100 to 50 hours, its only significant effect is on \( h^* \) (Table 3) which changes from 1.41 to 1.02 hours. For an increase in \( \lambda \) to 0.03, \( n^* \) and \( k^* \) are slightly affected and \( h^* \) drops to 0.78 hours. The optimum loss-costs for the three examples increase from \$401.38\) to \$694.77\) to \$959.47.\)

The effect of the value of \( \lambda \) on the loss-cost curves is evaluated by computing the quantity PIL for Examples 1, 2, and 3. The results are given in Table 5 and are plotted in Figure 4. We note that a change in the value of \( \lambda \) has only a moderate effect on the flexibility of the choice of the sample size as compared with that of \( \varepsilon \). In other words, when deviating from the optimum sample size, a low or a high value of \( \lambda \) has practically no effect on the percentage increase in loss-cost.

8. CONCLUSION

(1) The general solution derived in this paper consists of an implicit equation in \( n \) and \( k \) (3.11) and an explicit equation in \( h \) (3.12). The loss-cost is calculated from equation (3.1).

(2) This solution is exact and free of limitations on the cost and risk factors as compared with Duncan's approximate method.
in $\lambda$ on the optimum design, we consider Examples 1, 2, and 3, with values of $\lambda = 0.01$, 0.02, and 0.03 respectively. If $\lambda$ increases from 0.01 to 0.02, i.e. if the average time for an assignable cause to occur reduces from 100 to 50 hours, its only significant effect is on $h^*$ (Table 3) which changes from 1.41 to 1.02 hours. For an increase in $\lambda$ to 0.03, $n^*$ and $k^*$ are slightly affected and $h^*$ drops to 0.78 hours. The optimum loss-costs for the three examples increase from $401.38$ to $694.77$ to $959.47$.

The effect of the value of $\lambda$ on the loss-cost curves is evaluated by computing the quantity $P_{IL}$ for Examples 1, 2, and 3. The results are given in Table 5 and are plotted in Figure 4. We note that a change in the value of $\lambda$ has only a moderate effect on the flexibility of the choice of the sample size as compared with that of $e$. In other words, when deviating from the optimum sample size, a low or a high value of $\lambda$ has practically no effect on the percentage increase in loss-cost.

8. CONCLUSION

(1) The general solution derived in this paper consists of an implicit equation in $n$ and $k$ (3.11) and an explicit equation in $h$ (3.12). The loss-cost is calculated from equation (3.1).

(2) This solution is exact and free of limitations on the cost and risk factors as compared with Duncan's approximate method.
GLOSSARY

A-Parameters

δ - Shift in the process mean is δσ'

1/λ - Average time required for an assignable cause to occur

V₀ - Average income per hour when process is under control

V₁ - Average income per hour at the new level

M - Equals V₀ - V₁

e - Rate at which the time between the taking of a sample and the plotting of a point on the X-chart increases with the sample size n

D - Average time taken to find an assignable cause

T - Cost of looking for an assignable cause when none exists

W - Cost of looking for an assignable cause when it exists

b - Cost per sample of sampling and plotting

c - Cost per unit of sampling, testing and computation

σ'' - Standard value for standard deviation

B-Variables

n - Sample size

k - Control limit factor for the control chart

h - Interval between samples measured in hours

P - Probability that an assignable cause will be detected
B-Variables (contd)

\( Q \) - \((1 - P)\)

\( \alpha \) - Probability of looking for an assignable cause when it does not exist

\( L \) - Loss-cost

\( \beta \) - Proportion of the time process will be in control in many repetitions

\( \gamma \) - Proportion of the time process will be out of control in many repetitions

\( \epsilon \) - Average number of times per hour that the process actually goes out of control.

C-Mathematical Expressions

\[ P = \int_{-\sqrt{\frac{1}{2}}/2}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{z^2}{2}\right) \, dz \]

\[ \alpha = 2 \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{z^2}{2}\right) \, dz \]

\[ \beta = \frac{1}{\lambda} \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) h + \text{en} + \text{D} \]

\[ \gamma = \frac{1}{\lambda} \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) h + \text{en} + \text{D} \]

\[ \epsilon = \frac{1}{\lambda} \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) h + \text{en} + \text{D} \]
B-Variables (contd)

Q = (1 - P)

α - Probability of looking for an assignable cause when it does not exist

L - Loss-cost

β - Proportion of the time process will be in control in many repetitions

γ - Proportion of the time process will be out of control in many repetitions

ε - Average number of times per hour that the process actually goes out of control.

C-Mathematical Expressions

\[ P = \int_{k - \delta \sqrt{n}}^{\infty} \frac{1}{2\pi}^{1/2} \exp \left( -\frac{z^2}{2} \right) \, dz \]

\[ \alpha = 2 \int_{k}^{\infty} \frac{1}{2\pi}^{1/2} \exp \left( -\frac{z^2}{2} \right) \, dz \]

\[ \beta = \frac{1/\lambda}{\frac{1}{\lambda} + \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12}} \cdot \frac{1}{h + \text{en} + D} \]

\[ \gamma = \frac{1}{\lambda} + \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) \cdot \frac{1}{h + \text{en} + D} \]

\[ \epsilon = \frac{1}{\lambda} + \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) \cdot \frac{1}{h + \text{en} + D} \]
GLOSSARY

A-Parameters

δ - Shift in the process mean is \( \delta \sigma \)

\( 1/\lambda \) - Average time required for an assignable cause to occur

\( V_0 \) - Average income per hour when process is under control

\( V_1 \) - Average income per hour at the new level

\( M \) - Equals \( V_0 - V_1 \)

e - Rate at which the time between the taking of a sample and the plotting of a point on the \( \bar{X} \)-chart increases with the sample size \( n \)

D - Average time taken to find an assignable cause

T - Cost of looking for an assignable cause when none exists

W - Cost of looking for an assignable cause when it exists

b - Cost per sample of sampling and plotting

c - Cost per unit of sampling, testing and computation

\( \sigma \) - Standard value for standard deviation

B-Variables

n - Sample size

k - Control limit factor for the control chart

h - Interval between samples measured in hours

P - Probability that an assignable cause will be detected
REFERENCES


APPENDIX - A

Derivation of Equations (3.5) and (3.6)

We have

\[ L = \frac{xM + \frac{aT}{h} + \lambda W}{(1 + x)} + \frac{(b + cn)}{h} \ldots \quad (3.1) \]

and the conditions to be satisfied are:

\[ \frac{\partial L}{\partial k} = 0 \quad \ldots \quad (3.3) \]

and

\[ \frac{\partial L}{\partial h} = 0 \quad \ldots \quad (3.4) \]

Now

\[ \frac{\partial L}{\partial k} = \frac{(1 + x) \left( M \frac{\partial x}{\partial k} + \frac{T}{h} \frac{\partial a}{\partial k} \right) - (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial k}}{2 (1 + x)} \]

and

\[ \frac{\partial L}{\partial h} = \frac{(1 + x) \left( M \frac{\partial x}{\partial h} - \frac{\partial a}{\partial k} \right) - (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial h}}{(1 + x)²} - \frac{(b+cn)}{h²} \]

Equating \[ \frac{\partial L}{\partial k} \] and \[ \frac{\partial L}{\partial h} \] to zero, we get:

\[ (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial k} = (1 + x) (M \frac{\partial x}{\partial h} - \frac{aT}{h²}) - \frac{\partial a}{\partial k} = 0 \]

and

\[ (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial h} = (1 + x)(M \frac{\partial x}{\partial h} - \frac{aT}{h²}) - \frac{(b+cn)}{h²} (1 + x)² = 0 \]
APPENDIX - A

Derivation of Equations (3.5) and (3.6)

We have

\[ L = \frac{xM + \frac{aT}{h} + \lambda W}{(1 + x)} + \frac{(b + cn)}{h} \quad \ldots \quad (3.1) \]

and the conditions to be satisfied are:

\[ \frac{\partial L}{\partial k} = 0 \quad \ldots \quad (3.3) \]

and

\[ \frac{\partial L}{\partial h} = 0 \quad \ldots \quad (3.4) \]

Now

\[ \frac{\partial L}{\partial k} = \frac{(1 + x) \left( M \frac{\partial x}{\partial k} + \frac{T}{h} \frac{\partial a}{\partial k} \right) - (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial k}}{(1 + x)^2} \]

and

\[ \frac{\partial L}{\partial h} = \frac{(1 + x) \left( M \frac{\partial x}{\partial h} - \frac{aT}{h^2} \right) - (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial h}}{(1 + x)^2} - \frac{(b + cn)}{h^2} \]

Equating \( \frac{\partial L}{\partial k} \) and \( \frac{\partial L}{\partial h} \) to zero, we get:

\[ (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial k} - (1 + x) \left( M \frac{\partial x}{\partial k} + \frac{T}{h} \frac{\partial a}{\partial k} \right) = 0 \]

and

\[ (xM + \frac{aT}{h} + \lambda W) \frac{\partial x}{\partial h} - (1 + x) \left( M \frac{\partial x}{\partial h} - \frac{aT}{h^2} \right) - \frac{(b + cn)(1 + x)^2}{h^2} = 0 \]
REFERENCES


or \( \left( \frac{\alpha T}{h} + \lambda W - M \right) \frac{\partial x}{\partial k} = \left( \frac{1 + x}{h} \right) T \frac{\partial \alpha}{\partial k} \) \hspace{1cm} (A.1)

and \( \left( \frac{\alpha T}{h} + \lambda W - M \right) \frac{\partial x}{\partial h} = \left( \frac{1 + x}{h^2} \right) aT + \frac{(b + cn)(1 + x)^2}{h^2} \) \hspace{1cm} (A.2)

We find the expressions for \( \frac{\partial x}{\partial k} \) and \( \frac{\partial x}{\partial h} \) from the definition of \( x \). Recall that

\[ x = \left( \frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12} \right) \lambda h + \lambda (en + D) \] \hspace{1cm} (3.2)

Therefore

\[ \frac{\partial x}{\partial k} = \lambda \left( - \frac{h}{p^2} \frac{\partial p}{\partial k} \right) \] \hspace{1cm} (A.3)

and \( \frac{\partial x}{\partial h} = \lambda \left( \frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12} \right) + \frac{\lambda h}{12} \) \hspace{1cm} (A.4)

Also

\[ p = \int_{k-\delta}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \, dz \]

and \( \alpha = 2 \int_{k}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \, dz \)

By differentiating under the integral sign, we get

\[ \frac{\partial p}{\partial k} = \frac{- \exp\left\{ - (k-\delta)\sqrt{n} \right\}^2/2}{\sqrt{2\pi}} \] \hspace{1cm} (A.5)

and \( \frac{\partial \alpha}{\partial k} = - \frac{2 \exp(- k^2/2)}{\sqrt{2\pi}} \) \hspace{1cm} (A.6)
By substituting the value of $\frac{\partial P}{\partial k}$ from (A.5) in (A.3),

$$\frac{\partial x}{\partial k} = \frac{\lambda h}{\sqrt{2\pi}} \frac{1}{p^2} \exp \left( -k^2/2 \right) \exp \left( -\frac{(k - \delta \sqrt{\pi})^2}{2} \right)$$

(A.7)

Equating the values of $\frac{\partial x}{\partial k}$ from (A.1) and (A.7) and substituting the value of $\frac{\partial \mu}{\partial k}$ from (A.6), we get:

$$- \frac{2(1+x)}{\hbar} \frac{T}{\sqrt{2\pi}} \exp(-k^2/2) \frac{1}{(\frac{aT}{h} + \lambda W - M)} = \frac{\lambda h}{p^2} \frac{1}{\sqrt{2\pi}} \exp(-k^2/2)$$

(A.8)

Equating $\frac{\partial x}{\partial h}$ from (A.2) and (A.4), we have:

$$\left\{ - \frac{(1+x) aT}{h^2} + \frac{(b+cn)(1+x)^2}{h^2} \left( \frac{1}{aT/h} + \lambda W - M \right) \right\} = \lambda \left\{ \left( \frac{1}{p} - \frac{1}{2} + \frac{h}{12} \right) + \frac{h}{12} \right\}$$

(A.9)

Rearranging equations (A.8) and (A.9) yields equations (3.5) and (3.6) respectively

$$\frac{\lambda}{2TP^2} \frac{\exp \left\{ -\frac{(k-\delta \sqrt{\pi})^2}{2} \right\}}{\exp (-k^2/2)} = \frac{(1+x)}{(M - \lambda W) h^2 - aTh}$$

(3.5)

and

$$\lambda^2 \frac{h}{6} + (1/P - 1/2) \lambda = \frac{(1+x) \left\{ aT + (1+x)(b+cn) \right\}}{(M - \lambda W) h^2 - aTh}$$

(3.6)
By substituting the value of $\frac{\partial P}{\partial k}$ from (A.5) in (A.3),

$$\frac{\partial x}{\partial k} = \frac{\lambda h}{\sqrt{2\pi} p^2} \exp \left\{ - (k - \delta \sqrt{n})^2 / 2 \right\} \quad (A.7)$$

Equating the values of $\frac{\partial X}{\partial k}$ from (A.1) and (A.7) and substituting the value of $\frac{\partial u}{\partial k}$ from (A.6), we get:

$$\frac{-2(1+x)T}{h} \frac{\exp(-k^2/2)}{\sqrt{2\pi}} \frac{1}{\left(\frac{aT}{h} + \lambda W - M\right)} = \frac{\lambda h}{p^2 \sqrt{2\pi}} \exp\left\{ -(k - \delta \sqrt{n})^2 / 2 \right\} \quad (A.8)$$

Equating $\frac{\partial X}{\partial h}$ from (A.2) and (A.4), we have:

$$\left\{ - \frac{(1 + x)\frac{aT}{h^2} + \frac{(b + cn)(1 + x)^2}{h^2}}{\frac{1}{\sqrt{2\pi}} \frac{aT}{h} + \lambda W - M} \right\} = \lambda \left\{ \frac{1}{p^2} - \frac{1}{2} + \frac{\lambda h}{12} + \frac{\lambda h}{12} \right\} \quad (A.9)$$

Rearranging equations (A.8) and (A.9) yields equations (3.5) and (3.6) respectively

$$\frac{\lambda}{2TP^2} \frac{\exp \left\{ -(k - \delta \sqrt{n})^2 / 2 \right\}}{\exp \left\{ - k^2 / 2 \right\}} = \frac{(1 + x)}{(M - \lambda W) h^2 - a Th} \quad (3.5)$$

and

$$\lambda^2 h/6 + (1/P - 1/2) \lambda = \frac{(1 + x) \left\{ \frac{aT}{h} + \frac{(1 + x)(b + cn)}{h^2} \right\}}{(M - \lambda W) h^2 - a Th} \quad (3.6)$$
or \[
\left( \frac{\alpha T}{h} + \lambda W - M \right) \frac{\partial x}{\partial k} = \frac{(1 + x) T}{h} \frac{\partial \alpha}{\partial k}\]

(A.1)

and \[
\left( \frac{\alpha T}{h} + \lambda W - M \right) \frac{\partial x}{\partial h} = -\frac{(1 + x) \alpha T}{h^2} + \frac{(b + cn)(1 + x)^2}{h^2}\]

(A.2)

We find the expressions for \( \frac{\partial x}{\partial k} \) and \( \frac{\partial x}{\partial h} \) from the definition of \( x \). Recall that

\[
x = \left( \frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12} \right) \lambda h + \lambda (en + D) \quad (3.2)
\]

Therefore

\[
\frac{\partial x}{\partial k} = \lambda \left( -\frac{h}{p^2} \frac{\partial p}{\partial k} \right) \quad (A.3)
\]

and \( \frac{\partial x}{\partial h} = \lambda \left( \frac{1}{p} - \frac{1}{2} + \frac{\lambda h}{12} \right) + \frac{\lambda h}{12} \quad (A.4) \)

Also

\[
P = \int_{k-\delta}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \, dz
\]

\[
\text{and} \quad \alpha = 2 \int_{k}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \, dz
\]

By differentiating under the integral sign, we get

\[
\frac{\partial P}{\partial k} = -\frac{\exp\left(-\frac{(k-\delta \sqrt{n})^2}{2}\right)}{\sqrt{2\pi}} \quad (A.5)
\]

and

\[
\frac{\partial \alpha}{\partial k} = -\frac{2 \exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \quad (A.6)
\]
## APPENDIX B

**Cost and Risk Factors and Parameters for 18 Examples**

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#3
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General Solution for the Economic Design of X-Charts Based on Duncan's Model.

4. Descriptive Notes

5. Author(s)

S. M. Wu and A. L. Groel

6. Report Date

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7a. Total No. of pages

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Washington, D. C.

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Notes:
- λ: Cost factor
- δ: Risk factor
- M: Base parameter
- e: Exponent parameter
- D: Parameter decrement
- T: Parameter time
- W: Parameter width
- b: Parameter baseline
- c: Parameter constant

Examples 1-12 refer to the same set of parameters, with variations in λ, δ, and M. Examples 13-25 are similar, with variations in e, D, T, W, b, and c.