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Model Building in Two-sphere via Gauss-Weierstrass
Kernel Smoothing and Its Application
to Cortical Analysis, Part II.

Moo K. Chung\textsuperscript{1,2}, Steve Robbins\textsuperscript{4}, Kim M. Dalton\textsuperscript{2}
Shubing Wang\textsuperscript{1}, Alan C. Evans\textsuperscript{4}, Richard J. Davidson\textsuperscript{2,3}

\textsuperscript{1}Department of Statistics, Biostatistics, and Medical Informatics
\textsuperscript{2}Waisman Laboratory for Brain Imaging and Behavior
\textsuperscript{3}Department of Psychology and Psychiatry
University of Wisconsin-Madison
Madison, WI 53706. USA
\textsuperscript{4}Montreal Neurological Institute, McGill University, Canada
mchung@stat.wisc.edu
Tensor-based Cortical Morphometry via Weighted Spherical Harmonic Representation

Moo K. Chung1,2, Steve Robbins4, Kim M. Dalton2
Shubing Wang1, Alan C. Evans3, Richard J. Davidson2,4
1Department of Statistics, Biostatistics, and Medical Informatics
2Wisconsin Laboratory for Brain Imaging and Behavior
3Department of Psychology and Psychiatry
University of Wisconsin-Madison
Madison, WI 53706. USA
4Montreal Neurological Institute, McGill University, Canada
mchung@stat.wisc.edu
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Abstract

We present a new tensor-based morphometric framework that quantifies cortical shape variations via the concept of local area element. The local area element is obtained from the Riemannian metric tensors, which are, in turn, obtained from the smooth functional parametrization of a triangle cortical mesh. For the smooth parametrization, we have developed a novel weighted spherical harmonic (SPHARM) representation. The weighted-SPHARM differs from the classical SPHARM in a regularizing cost function. The classical SPHARM is the asymptotic limit of the weighted-SPHARM. Further, for a specific choice of weights, the weighted-SPHARM is shown to be the finite least squares approximation to the solution of an isotropic heat diffusion on a unit sphere. The main aims of this paper are to present a theoretical framework for the weighted-SPHARM, and to show how it can be used in the tensor-based morphometry. As an illustration, the methodology has been applied in the problem of detecting abnormal cortical regions in a clinical population.

1 Introduction

In many previous cortical morphometric studies, cortical thickness have been mainly used to quantify cortical shape variations in a population [12, 15, 16]. The cortical thickness measures the amount of gray matter in vertical direction on a cortical surface. We present a new tensor-based morphometry (TBM) that quantifies the amount of gray matter along the tangential direction of the cortex by computing the local area element. The local area element is obtained from the Riemannian metric tensors, which are, in turn, computed from the smooth functional parametrization of a cortical mesh. For this purpose, we present a novel weighted spherical harmonic (SPHARM) representation that differs from the classical SPHARM [9, 19] in regularizing a cost function. Unlike the classical SPHARM, we weight the measurement such that the closer measurements are weighted more. The weighted-SPHARM is related to both the classical SPHARM and an isotropic heat diffusion on a unit sphere as asymptotic limits.

Let us overview some of previous studies that are related to our study. Gerig et al. (2001) used the mean squared distance (MSD) of SPHARM coefficients in quantifying ventricle surface shape in a twin study [9]. The distance based metrics widely used in deformation-based morphometry [1] do not directly quantify the amount of tissue growth and atrophy. For directly measuring the amount of tissue volume, the Jacobian determinant of the deformation field is a better metric [3]. Our local area element is the differential geometric generalization of the Jacobian determinant in Riemannian manifolds. So the area element will be able to quantify the cortical tissue growth/atrophy directly.

Shen et al. (2004) used the principal component analysis technique on the SPHARM coefficients of schizophrenic hippocampal surface in reducing the data dimension [19]. Then they performed the linear discriminant analysis and support vector machines in surface classification. In a related work, Gu et al. (2004) presented the SPHARM representation as a surface compression technique, where the main geometric features are encoded in the low degree spherical harmonics, while the noise will
be in the high degree spherical harmonics [10]. It will be shown that our weighted-SPHARM more penalizes the high degree spherical harmonics compared to the classical SPHARM.

Bulow (2004) used spherical harmonics in developing an isotropic heat diffusion via Fourier transform on a unit sphere as form of hierarchical surface representation [2]. We will show that our weighted-SPHARM representation is related to Bulow’s heat diffusion asymptotically.

Many SPHARM literatures [2, 9, 10, 19] use the both real- and imaginary-valued spherical harmonics. However, the coefficients of imaginary-valued spherical harmonic basis do not serve any purpose in SPHARM representation other than the mathematical simplicity of manipulating them. In our study, we will only use real-valued spherical harmonics with different normalizing constants. This has the effect of decreasing the number of coefficients to be estimated by half.

Once the differentiable smooth parametrization of the cortex is established by the weighted-SPHARM, we can compute the Riemannian metric tensors and local area element. Many previous differential geometric cortical modeling is based on locally fitting a quadratic polynomial [6, 8]. The SPHARM-based global parametrization tend to be computationally expensive compared to the local quadratic polynomial based parametrization while providing more accuracy and flexibility for hierarchical representation.

2 Preliminary

In this section, we introduce mathematical notations and basic concepts in SPHARM that are needed in describing the weighted-SPHARM.

2.1 Surface parametrization

Let $\mathcal{M}$ and $S^2$ be a cortical surface and a unit sphere respectively. $\mathcal{M}$ and $S^2$ are realized as meshes with more than 80,000 triangle elements. It is natural to assume the cortical surface to be a smooth 2-dimensional Riemannian manifold parameterized by two parameters [7]. This parametrization is constructed in the following way. A point $p = (x, y, z) \in \mathcal{M}$ is mapped onto $u = (u_1, u_2, u_3) \in S^2$ via a deformable surface algorithm that preserves anatomical homology and the topological connectivity of meshes (Figure 1).

Let $U$ be the inverse mapping from $S^2$ to $\mathcal{M}$. Point $u = (u_1, u_2, u_3) \in S^2$ is parameterized by the spherical coordinates:

$$(u_1, u_2, u_3) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

Figure 1: Cortical manifold $\mathcal{M}$ (left) is mapped onto unit sphere $S^2$ (right) via a deformable surface algorithm that preserves anatomical homology [15]. For the visualization purpose, the mean curvature was computed and thresholded to better represent sulci and gyri.

with $(\theta, \varphi) \in \mathcal{N} = [0, \pi] \times [0, 2\pi)$. This mapping will be denoted as $X$, i.e., $X : \mathcal{N} \rightarrow S^2$. Then we have composite mapping $Z = U \circ X : \mathcal{N} \rightarrow \mathcal{M}$. $Z$ is a 3D vector and it will be stochastically modeled as

$$Z(\theta, \varphi) = \nu(\theta, \varphi) + \epsilon(\theta, \varphi), \quad (1)$$

where $\nu$ is unknown true differentiable parametrization and $\epsilon$ is a random vector field on unit sphere. The computation of the Riemannian metric tensors and local area element require estimating differentiable function $\nu$.

2.2 Spherical harmonic representation

The basis functions on the unit sphere are given as the eigenfunctions satisfying $\Delta f + \lambda f = 0$, where $\Delta$ is the spherical Laplacian:

$$\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

There are $2l+1$ eigenfunctions, denoted as $Y_{lm}(-l \leq m \leq l)$, corresponding to the same eigenvalue $\lambda = l(l+1)$. $Y_{lm}$ is called the spherical harmonic of degree $l$ and order $m$ [5, 21]. For the completeness of the exposition, we write the explicit form of the $2l+1$ spherical harmonics of degree $l$ as

$$Y_{lm} = \begin{cases} 
- \frac{c_{lm} P_l^{|m|}(\cos \theta) \sin(|m| \varphi)}{\sqrt{\pi} P_l^{|m|}(\cos \theta)}, & -l \leq m \leq -1, \\
- \frac{c_{lm} P_l^{|m|}(\cos \theta)}{\sqrt{2\pi} P_l^{|m|}(\cos \theta)}, & m = 0, \\
c_{lm} P_l^{|m|}(\cos \theta) \cos(|m| \varphi), & 1 \leq m \leq l,
\end{cases}$$

where $c_{lm} = \sqrt{\frac{2l+1}{2\pi} \frac{(l-|m|)!}{(l+|m|)!}}$ and $P_l^m$ is the associated Legendre polynomials of order $m$. Unlike many previous
SPHARM literatures [2, 9, 10, 19] that used the complex-valued spherical harmonics, we use only real-valued spherical harmonics with different normalizing constants since they are more convenient for a real-valued stochastic model (1).

For \( f, h \in L^2(S^2) \), the space of square integrable functions in \( S^2 \), the inner product is defined as

\[
\langle f, h \rangle = \int_0^{2\pi} \int_0^\pi f(\theta, \varphi) h(\theta, \varphi) \sin \theta \ d\theta \ d\varphi,
\]

where Lebesgue measure \( d\mu(\theta, \varphi) = \sin \theta \ d\theta \ d\varphi \). Consider subspace

\[
\mathcal{H}_k = \left\{ \sum_{l=0}^k \sum_{m=-l}^l \beta_l Y_{lm} : \beta_l \in \mathbb{R} \right\} \subset L^2(S^2),
\]

which is spanned by up to \( k \)-th degree spherical harmonics. Then the least squares estimation, denoted as \( \hat{f} \), of \( f \in L^2(S^2) \) in the subspace \( \mathcal{H}_k \) is given by

\[
\hat{f}(p) = \sum_{l=0}^k \sum_{m=-l}^l \langle f, Y_{lm} \rangle Y_{lm}(p).
\]

This can be stated as the following theorem.

**Theorem 1.**

\[
\sum_{l=0}^k \sum_{m=-l}^l \langle f, Y_{lm} \rangle Y_{lm} = \arg \min_{h \in \mathcal{H}_k} \int_{S^2} [f(q) - h(q)]^2 \ d\mu(q).
\]

This theorem is a well known result and mainly refered as the generalized Fourier series expansion. This is the basis of the classical SPHARM representation of closed anatomical boundaries [9, 10, 19].

### 3 Weighted-SPHARM

#### 3.1 Basic theory

The classical SPHARM is only one possible representation of functional data measured on the unit sphere. We will present a more general representation technique in the framework of a local kernel regression [11]. We will call this technique as the weighted-SPHARM since the coefficients of SPHARM are additionally weighted by the eigenvalues of a kernel. It will be shown that the classical SPHARM is asymptotically related to the weighted-SPHARM.

First, we start with the spectral representation of positive definite kernel in \( S^2 \). Any positive definite kernel \( K(p, q) \) in \( S^2 \) can be represented as

\[
K(p, q) = \sum_{l=0}^\infty \sum_{m=-l}^l \lambda_{lm} Y_{lm}(p) Y_{lm}(q),
\]

where eigenvalues \( \lambda_{00} \geq \lambda_{11} \geq \lambda_{22} \geq \cdots \geq 0 \) satisfy

\[
\int_{S^2} K(p, q) Y_{lm}(q) \ d\mu(q) = \lambda_{lm} Y_{lm}(p).
\]

This is the special case of the Mercer’s theorem [5]. Without loss of generality, we assume the kernel is normalized in such a way that

\[
\int_{S^2} K(p, q) \ d\mu(q) = 1.
\]

At each fixed point \( p \), smooth representation \( h \) of functional data \( f \) is searched in the subspace \( \mathcal{H}_k \) that minimizes the integral of the weighted squared distance between \( f \) and \( h \). This is formulated as the following regularization problem:

\[
\min_{h \in \mathcal{H}_k} \int_{S^2} K(p, q) [f(q) - h(p)]^2 \ d\mu(q).
\]
See Figure 2 for the schematic comparison of the classical SPHARM and the weighted-SPHARM. The minimizer of (4) is given by the following theorem.

**Theorem 2.**
\[
\sum_{l=0}^{k} \sum_{m=-l}^{l} \lambda_{lm}(f, Y_{lm}) Y_{lm} = \arg \min_{h \in \mathcal{H}_k} \int_{S^2} K(p, q)[f(q) - h(p)]^2 \, d\mu(q).
\]

**Proof.** Let
\[
h(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p) \in \mathcal{H}_k.
\]
The integral can be written as
\[
I(\beta_{l0}, \beta_{l1}, \beta_{l2}, \ldots, \beta_{lk}) = \int_{S^2} K(p, q)[f(q) - \sum_{l=0}^{k} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p)]^2 \, d\mu(q).
\]
Since the functional \(I\) is quadratic in \(\beta_{lm}\), the minimum exists and it is obtained when
\[
\frac{\partial I}{\partial \beta_{l'm'}} = 0 \quad \text{for all } l' \text{ and } m'.
\]
Then solving equation (5) with equation (3), we have
\[
Y_{l'm'}(p) \int_{S^2} K(p, q)f(q) \, d\mu(q) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \lambda_{lm}(f, Y_{lm}) Y_{lm}(p) Y_{l'm'}(p).
\]
Integrate the both sides of the above equation with respect to measure \(\mu(p)\). Then using the othonormal condition
\[
\int_{S^2} Y_{ij}(p) Y_{lm}(p) \, d\mu(p) = \begin{cases} 1, & i = l, j = m, \\ 0, & \text{otherwise}, \end{cases}
\]
we obtain
\[
\beta_{l'm'} = \int_{S^2} f(q) \, d\mu(q) \int_{S^2} Y_{l'm'}(p) K(p, q) \, d\mu(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \lambda_{lm} \int_{S^2} f(q) Y_{lm}(q) \, d\mu(q)
\times \int_{S^2} Y_{lm}(p) Y_{l'm'}(p) \, d\mu(p)
\times \lambda_{lm} \int_{S^2} f(q) Y_{l'm'}(q) \, d\mu(q).
\]
This proves the statement.

Now we show what happens as the dimension of \(\mathcal{H}_k\) increases. Define kernel smoothing as the integral convolution
\[
K * f(p) = \int_{S^2} f(q) K(p, q) \, d\mu(q).
\]
Then it can be shown that the weighted-SPHARM converges to kernel smoothing (8) as the dimension of subspace \(\mathcal{H}_k\) increases. This can be stated differently as

**Theorem 3.**
\[
K * f(p) = \arg \min_{h \in L^1(S^2)} \int_{S^2} K(p, q)[f(q) - h(p)]^2 \, d\mu(q).
\]

**Proof.** The weighted-SPHARM representation can be rearranged as
\[
\sum_{l=0}^{k} \sum_{m=-l}^{l} \lambda_{lm}(f, Y_{lm}) Y_{lm}(p) \int_{S^2} f(q) \sum_{l=0}^{k} \sum_{m=-l}^{l} \lambda_{lm}(f, Y_{lm}) Y_{lm}(q) \, d\mu(q)
\]
\[
\to \int_{S^2} f(q) K(p, q) \, d\mu(q) \quad \text{as } k \to \infty.
\]
The last line is from equation (2). On the other hand, from the completeness of Hilbert space \(L^2(S^2)\),
\[
\lim_{k \to \infty} \arg \min_{h \in \mathcal{H}_k} \int_{S^2} K(p, q)[f(q) - h(p)]^2 \, d\mu(q) = \arg \min_{h \in L^1(S^2)} \int_{S^2} K(p, q)[f(q) - h(p)]^2 \, d\mu(q).
\]
This proves the statement. Theorem 3 connects the weighted-SPHARM to kernel smoothing as the asymptotic limit.

We briefly explain the numerical implementation issues. We only need to numerically compute the Fourier coefficients \((f, Y_{lm})\) in the weighted-SPHARM. The eigenvalues \(\lambda_{lm}\) are given analytically from an analytical kernel. The computation for the Fourier coefficients are based on the direct numerical integration over high resolution triangle meshes with more than 80,000 triangles and the average inter-vertex distance of 0.0189 mm. The accuracy of the weighted-SPHARM is only restricted to the mesh resolution and the Riemann sum approximation should converge to the integral as the mesh resolution increases.

Let \(n\) be the total number of nodes in the mesh. Assume triangle elements \(T_{k_1}, \ldots, T_{k_n}\) are adjacent to each other at node \(q_k\) (1 \(\leq k \leq n\)). Then the Fourier coefficients \((f, Y_{lm})\) is approximated as a limit of the Riemann sum over triangle elements:
\[
\int_{S^2} f(q) Y_{lm}(q) \, d\mu(q) = \frac{1}{3} \sum_{k=1}^{n} \sum_{i=1}^{k-1} f(q_k) Y_{lm}(q_k) |T_k|.
\]
The weighted-SPHARM is constructed by iteratively adding each term in Theorem 2.

3.2 Gauss-Weistrass kernel and heat flow

For the choice of eigenvalues

$$\lambda_{lm} = e^{-l(l+1)\sigma},$$  \hspace{1cm} (9)

the corresponding kernel is called the Gauss-Weistrass kernel and it will be denoted as

$$K_{\sigma}(p, q) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} Y_{lm}(p) Y_{lm}(q).$$  \hspace{1cm} (10)

The subscript $\sigma$ is introduced to indicate the dependence of the additional parameter. As $\sigma \to \infty$, $\lambda_{lm} \to 1$ and the weighted-SPHARM representation converges to the classical SPHARM representation. It is interesting to note that even though the regularizing cost functions are different in Theorem 1 and Theorem 2, they are related asymptotically.

Another interesting property is observed by noting that $K_{\sigma} * f$ is the unique solution to the following isotropic heat diffusion

$$\frac{\partial g}{\partial t} = \Delta g, \quad g(p, t = 0) = f(p)$$ \hspace{1cm} (11)

at time $t = \sigma^2/2$ [4, 18]. This is easily seen from the fact that $K_{\sigma}$ is the Green’s function of equation (11). From this property combined with Theorem 3, we conclude that the weighted-SPHARM is the finite approximation to the isotropic heat flow in $S^2$.

We have compared the numerical implementation of the weighted-SPHARM result against the analytical solution of (11). Let $f = e^{l(l+1)Y_{lm}(\theta, \varphi)}$ be an analytic test function. Then $K_{\sigma} * f$ can be written as

$$e^{l(l+1)} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} Y_{lm}(p) Y_{lm}(q) d\mu(q)$$

$$= e^{l(l+1)} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} Y_{lm}(p) \delta_{ll'} \delta_{mm'} = Y_{lm}(p),$$

where $\delta_{ll'}$ is the Kroneker’s delta. The Table 1 shows the comparative result for $l' = 20$ and selective $m'$ with $\sigma = 0.01$ and degree $k = 20$. The third column shows the numerical computation of integral $\int_{S^2} Y_{lm}^2(p) \, dp = 1$ showing the accuracy up to 3 decimal places. This shows our Riemann sum approximation provides sufficiently good accuracy, which depends on the mesh resolution. The fourth column shows the average difference between the weighted-SPHARM and the heat diffusion.

### 3.3 Riemannian metric tensor estimation

The weighted-SPHARM estimation $\hat{\nu}$ of the unknown true parameterization $\mu$ in equation (1) is given by

$$\hat{\nu} = \sum_{l=0}^{k} \lambda_{lm} \langle Z, Y_{lm} \rangle Y_{lm}.$$  \hspace{1cm} (12)

For this study, we used eigenvalue (9) corresponding to the Gauss-Weistrass kernel. The estimation of the Riemannian metric tensors requires partial derivatives of $\hat{\nu}$. Denoting partial differential operators as $\partial_1 = \partial_\theta$ and $\partial_2 = \partial_\varphi$, we have derivative estimations

$$\partial_1 \hat{\nu} = \sum_{l=0}^{k} \lambda_{lm} \langle Z, Y_{lm} \rangle \partial_1 Y_{lm}(\theta, \varphi),$$

$$\partial_2 \hat{\nu} = \sum_{l=0}^{k} \lambda_{lm} \langle Z, Y_{lm} \rangle \partial_2 Y_{lm}(\theta, \varphi).$$

<table>
<thead>
<tr>
<th>$l'$</th>
<th>$m'$</th>
<th>integral</th>
<th>difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>4</td>
<td>1.0001</td>
<td>9.7029 · 10^{-5}</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>0.9999</td>
<td>1.6212 · 10^{-4}</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>0.9999</td>
<td>-1.1174 · 10^{-4}</td>
</tr>
</tbody>
</table>

Table 1: Numerical accuracy of the weighted-SPHARM with $\sigma = 0.01$ for degree 20, and order 4, 10 and 20. The third column checks if $\langle Y_{lm}, Y_{lm'} \rangle = 1$. The last column shows the average difference between the weighted-SPHARM and the expected heat diffusion. In each case...
Figure 4: Riemannian metric tensor estimation. The metric tensors $g_{ij}$ are estimated by differentiating the weighted-SPHARM representation. Afterwards the local area element $\sqrt{\det g}$ is computed. The local area element measures the amount of area expansion and shrinking with respect to $S^2$.

The partial derivatives of spherical harmonics are computed iteratively. The associated Legendre polynomials in the spherical harmonic basis are defined as

$$P_l^m(\cos \theta) = \sin^m \theta \frac{d^m}{d\theta^m} P_l(x) \bigg|_{x=\cos \theta},$$

where $P_l(x)$ is the Legendre polynomials defined in $(-1, 1)$ with $P_0(x) = 1$ and $P_1(x) = x$. Then for $0 \leq m \leq l - 1,$

$$\partial_\theta P_l^m(\cos \theta) = \sin^{m-1} \theta \cos \theta \frac{d^m}{d\theta^m} P_l(x) \bigg|_{x=\cos \theta}$$

$$- \sin^{m+1} \theta \frac{d^{m+1}}{d\theta^{m+1}} P_l(x) \bigg|_{x=\cos \theta}$$

$$= m \cot \theta P_l^m(\cos \theta) - P_l^{m+1}(\cos \theta).$$

For $m = l$, since $P_l$ is the $l$-th order polynomial, the second term vanishes. A similar recursive relationship for an alternate definition for the associated Legendre polynomial is given in [13]. Based on this iterative relation, we can compute the partial derivatives

$$\partial_\phi Y_{lm} = \begin{cases} 
    c_{lm} \partial_\theta P_l^m(\cos \theta) \cos(|m|\varphi), & -l \leq m \leq -1, \\
    c_{lm} \partial_\theta P_l^m(\cos \theta), & m = 0, \\
    c_{lm} \partial_\theta P_l^m(\cos \theta) \sin(|m|\varphi), & 1 \leq m \leq l
\end{cases}$$

and

$$\partial_\varphi Y_{lm} = \begin{cases} 
    |m|c_{lm} P_l^m(\cos \theta) \cos(|m|\varphi), & -l \leq m \leq -1, \\
    0, & m = 0, \\
    -|m|c_{lm} P_l^m(\cos \theta) \sin(|m|\varphi), & 1 \leq m \leq l.
\end{cases}$$

Then the Riemannian metric tensors are estimated as $g = (g_{ij}) = (\partial_\phi \tilde{g}, \partial_\phi \tilde{g})$ and the area element

$$G(\theta, \varphi) = \sqrt{\det g} = \sqrt{g_{11} \cdot g_{22} - g_{12}^2}.$$

The area element measures the transformed area in $M$ of the unit square of the parameterized space $N$ via mapping $\nu$.

4 Statistical inference in $S^2$

For the $i$-th subject ($1 \leq i \leq m$), we denote the cortical manifold as $M_i$ and its area element as $G_i(\theta, \varphi)$.

The area element is influenced by the global brain size. If we enlarge the cortical coordinates by the factor of $r$, the area element changes by the factor of $r^2$. So it is necessary to normalize $G_i$ such that it is invariant under scaling. The affine scale invariant area element is given by

$$\overline{G}_i(\theta, \varphi) = \frac{4\pi G_i(\theta, \varphi)}{\mu(M_i)},$$

where $\mu(M_i)$ is the total cortical area. If we enlarge the the cortical coordinates by the factor of $r$, $\mu(M_i)$ increases by the factor of $r^2$ making $G_i$ invariant under affine scaling. The constant $4\pi$ is multiplied so that the normalization is
with respect to the total surface area of $S^2$. Then we have the following general linear model (GLM):

$$
\hat{G}_i(\theta, \varphi) = \alpha_0 + \alpha_1 \cdot \text{age} + \alpha_2(\theta, \varphi) \cdot \text{group} + \epsilon(\theta, \varphi),
$$

where $\epsilon$ is a mean zero Gaussian random field. age and group are the age and a categorical dummy variable (0 for autism and 1 for control) respectively for subject $i$. Then we test if there is any group difference in the local area element measurement by testing

$$
H_0 : \alpha_2(\theta, \varphi) = 0 \text{ for all } \theta \text{ and } \varphi,
$$

vs.

$$
H_1 : \alpha_2(\theta, \varphi) \neq 0 \text{ for some } \theta \text{ and } \varphi.
$$

At each point $(\theta, \varphi)$, a $F$-statistic with 1 and $n-3$ degrees of freedom, denoted as $F(\theta, \varphi)$ is used as a test statistic. The $F$-statistic is constructed as a ratio of the residual sum of error of model fit of $H_0$ and $H_1$. Since we need to perform the test at every $(\theta, \varphi)$, this becomes a multiple comparison problem. We used the random field theory [20, 22] based thresholding to determine the statistical significance.

The probability of obtaining false positives ($\alpha$-level) for the one sided alternate hypothesis in given by

$$
P(\sup F(\theta, \varphi) \geq F_\alpha) = \sum_{i=0}^{2} \frac{L_i(S^2)}{\text{FWHM}_i} \rho_i(y),
$$

where $L_i$ is the $i$-th Lipschitz-Killing curvature or Minkowski functional [20], and $\rho_i$ is the $i$-dimensional EC-density [22]. FWHM denotes the full width of the half maximum of smoothing kernel $K_\sigma$ used in the weighted spherical harmonic representation. For the unit sphere, the Lipschitz-Killing curvatures are

$$
L_0(S^2) = 2, \quad L_1(S^2) = 0, \quad \text{and } L_2(S^2) = 2\pi.
$$

The EC-densities are

$$
\rho_0(y) = \int_y^\infty \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m-1)!/2/\Gamma\left(m-1/2\right)} \left(1 + \frac{y^2}{m-1}\right)^{-m/2} dy,
$$

$$
\rho_2(y) = \frac{4 \ln 2}{\pi} \frac{\Gamma\left(\frac{m+1}{2}\right) y}{(\pi)^{3/2}} \left(1 + \frac{y^2}{m-1}\right)^{-m/2}.
$$

4.1 Computing FWHM

The computation for the FWHM of the Gauss-Weissstrass kernel in $S^2$ is not trivial due to the fact there is no known close form expression for the FWHM as a function of $\sigma$. So the FWHM is computed numerically.

The Gauss-Weissstrass kernel can be simplified from equation (10), via the harmonic addition theorem [21], as

$$
K_\sigma(p, q) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} e^{-l(l+1)\sigma} P_l^0(\cos \vartheta),
$$

where $\vartheta$ is the angle between $p$ and $q$. Using the vector inner product $\cdot$, the angle can be written as $\cos \vartheta = p \cdot q$. The maximum of the Gauss-Weissstrass kernel is obtained when $\vartheta = 0$ and it is given by

$$
\sum_{l=0}^{k} \frac{2l+1}{4\pi} e^{-l(l+1)\sigma}.
$$

Now we fix $\varphi = 0$ and let $p$ be the north pole, i.e. $p = (0, 0, 1)$. By varying $q = (\sin \theta, 0, \cos \theta)$ for $0 \leq \theta \leq \cos^{-1}(p \cdot q) \leq \pi$, we have $Y_{lm} = 0$ if $m \neq 0$. Note $P_l^0(1) = 1$ for all $l$. Then we solve numerically for $\theta$ in

$$
\sum_{l=0}^{k} \frac{2l+1}{4\pi} e^{-l(l+1)\sigma} = \sum_{l=0}^{k} \frac{2l+1}{4\pi} e^{-l(l+1)\sigma} P_l^0(\cos \vartheta).
$$

The FWHM is then $2\theta$. Table 2 in Figure 5 shows the nonlinear relationship between $\sigma$ and the corresponding FWHM for $k = 20$.

5 Application to autism study

Three Tesla $T_1$-weighted MR scans were acquired for 16 autistic and 12 control right handed males. 16 autistic subjects were diagnosed with high functioning autism. The average ages are $17.1 \pm 2.8$ and $16.1 \pm 4.5$ for control and autistic group respectively. Image intensity nonuniformity was corrected using nonparametric nonuniform intensity normalization method and then the image was spatially normalized into the Montreal neurological institute (MNI) stereotopic space using a global affine transformation. Afterwards, an automatic tissue-segmentation algorithm based on a supervised artificial neural network classifier was used to classify each voxel as cerebrospinal fluid (CSF), gray matter, or white matter [14]. Triangular meshes for outer cortical surfaces were obtained by a deformable surface algorithm [13]. The mesh starts as an ellipsoid located outside the brain and is shrunk to match the cortical boundary. By performing an affine transform on this ellipsoid, we obtain $S^2$ mesh, which is used in the weighted-SPHARM.

The segmented cortical meshes are normalized via a nonlinear surface-to-surface registration [17]. Cortical curvatures of two surfaces are mapped onto the sphere and they are aligned by solving a regularization problem that tries to minimize the discrepancy between two curvatures while maximizing the smoothness of the alignment in such a way that the pattern of gyral ridges are matched smoothly. This regularization mechanism produces a smooth deformation field, with very little folding. The deformation field is parameterized using a triangulated mesh and the algorithm proceeds in a coarse-to-fine manner, with four levels of mesh resolution. The surface-to-surface registration is
6 Conclusions and discussions

In this paper, we presented a theoretical framework for the weighted-SPHARM and its application in TBM. The weighted-SPHARM is used as a differentiable parametrization of the cortex. This enables us to compute the Riemannian metric tensors and local area element. The local area element is used to determine the statistical significance of the abnormal cortical tissue expansion/shrinking for a clinical population.

The weighted-SPHARM is a very flexible functional estimation technique for scalar and vector data defined in $S^2$. It was shown that the solution to the isotropic heat diffusion in $S^2$ is the asymptotic limit of the weighted-SPHARM for the choice of the Gauss-Weierstrass kernel. We can extend this argument further. By choosing the Green’s function of a particular partial differential equation (PDE) as the kernel in the weighted-SPHARM, we can construct the finite least squares estimation of the solution of the PDE without numerically solving PDE using the finite element technique. This should serve as a springboard for investigating the wide variety of PDE-based data smoothing techniques in the local polynomial regression framework [11].

References


