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THE EFFECT OF EXPERIMENTAL ERROR
ON THE DETERMINATION OF THE
OPTIMUM METAL-CUTTING CONDITIONS

by
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INTRODUCTION

It has been shown in previous studies of machining economics that the determination of the optimum cutting conditions depends on the cost and time parameters, and the parameters in the tool-life equation \((1, 4, 8)\). These investigations have assumed that for a given set of operating conditions the parameters in the tool-life equation are constants. In reality, the parameters in the tool-life equation are empirical estimates that are subject to uncertainty because of the experimental error in tool-life testing. Consequently, the theoretically optimum cutting conditions are also affected by the uncertainty in these estimates. The effect of this uncertainty is investigated assuming that the optimizing criterion is minimum cost.

The unit-cost model which describes the average cost to produce a workpiece by means of the basic turning operation is

\[
C_u = c_o t_m + \frac{t_m}{T} (c_o t_c + c_t) + c_o t_h
\]

(1)

where

- \(c_o\) is the cost of operating time
- \(c_t\) is the tool cost
- \(t_m\) is the time to machine a workpiece
- \(t_h\) is the handling time
- \(T\) is the tool life

Based on this model and Taylor's tool-life equation, \(VT^2 = C\), where \(V\) is the cutting speed, and \(n\) and \(C\) are the parameters to be estimated, the theoretical cutting speed for minimum cost is

\[
V_{\text{min}} = \left[ \frac{C}{\left( \frac{1}{n} - 1 \right) \left( t_c + \frac{t_h}{c_o} \right)} \right]^{\frac{n}{2}} = \left[ \frac{C}{\left( \frac{1}{n} - 1 \right) t_e} \right]^{\frac{n}{2}}
\]

(2)

* Underlined numbers in parentheses designate references at the end of the paper
where

\[ t_e = t_c + \frac{c}{c_o} \]

Accordingly, the minimum-cost cutting speed is subject to uncertainty due to the experimental error in estimating \( n \) and \( C_e \), and therefore \( V_{\min} \) is not uniquely defined but lies within some probable interval of cutting speeds. This probable interval for \( V_{\min} \) depends on the magnitude of the experimental error in tool-life testing and the cost-time parameter, \( t_e \). In addition, the experimental range of cutting conditions in tool-life testing also influences the probable interval for \( V_{\min} \). The interrelationships between the experimental error, the cost-time parameters, and the testing range further complicates the analysis.

An estimate of the experimental error is provided by the residual sum of squares from a least squares fit. The precision of the parameter-estimates is indicated by confidence intervals, which are used to determine the effect of experimental error. An example illustrates the effect of the uncertainty in \( n \) and \( C_e \), and a decision rule based on the minimax principle is explained to show how this principle can be applied in the determination of \( V_{\min} \) under uncertainty. The influence of the cost-time parameter \( t_e \) is considered, and the study is expanded to include feed as a variable.

It should be noted that the primary objective is assumed to be minimum cost rather than maximum profit in order to simplify the analysis of the effect of experimental error (8). In addition, the work in this paper is based on a logarithmic transformation of Taylor's tool-life equation, although it should
be recognized that the best transformation for fitting tool-life data can be used in future work (7).

The Tool-Life Equation and the Estimated Error Variance $S^2$

To show the effect of experimental error on the determination of $V_{min}$ it is convenient to work with both the conventional form ($VT^n = C$) and the logarithmic form of Taylor's equation. A logarithmic transformation of Taylor's equation is of the form

$$\hat{y} = \ln T = b_0 + b_1 \ln V = b_0 x_o + b_1 x_1$$  \hspace{1cm} (3)

where $y$ is the predicted value of tool life on a logarithmic scale, $x_o$ is unity at all levels, $x_1$ is the cutting speed on a logarithmic scale, and $b_0$ and $b_1$ are the least squares estimates of the parameters $\beta_0$ and $\beta_1$ from the postulated model

$$E(y) = E(\ln T) = \beta_0 x_o + \beta_1 x_1$$  \hspace{1cm} (4)

where $E(y)$ is the expected value of the average tool life on a logarithmic scale. An equation equivalent to (3) is

$$\hat{y} = \bar{y} + b_1 (x_1 - \bar{x}_1) = (\bar{y} - b_1 \bar{x}_1) + b_1 x_1$$  \hspace{1cm} (5)

which represents the fitted line in terms of the slope ($b_1$) and the centroid ($\bar{x}_1, \bar{y}$), where $\bar{x}_1$ is the average of the cutting speeds and $\bar{y}$ is the average of the tool-life observations. Thus,

$$b_0 = \bar{y} - b_1 \bar{x}_1$$  \hspace{1cm} (6)

and $b_0$ and is correlated with $b_1$.

The relationships for converting from $b_0$ and $b_1$ in the linear form to the estimates $C$ and $n$ in the conventional form of Taylor's tool-life equation
are easily defined by noting that a logarithmic transformation of $VT^n = C$ is of the form

$$\ln T = \frac{\ln C}{n} - \frac{1}{n} \ln V$$

(7)

Comparing equations (3) and (7), the conversion relationships are

$$n = -\frac{1}{b_1}$$

(8)

and

$$C = \exp \{ n b_o \} = \exp \{- \frac{b_o}{b_1} \}$$

(9)

To determine the confidence intervals an estimate of the experimental error is needed. The magnitude of the experimental error is estimated by the error variance,

$$s^2 = \frac{\sum_{i=1}^{N} (y_i - \hat{y}_i)^2}{N - q}$$

(10)

where $y_i$ is the $i$-th observation of tool-life on a logarithmic scale, $N$ is the number of observations, and $q$ is the number of parameters to be estimated.

An Example

This example is based on a subset of the carbide tool-life data shown in Appendix A. The postulated model used to fit the data is given by equation (4), and equations (8) and (9) are used to convert from the least squares estimates $b_o$ and $b_1$ to the estimates $n$ and $C$. The least squares estimates are found from $b = (X'X)^{-1} X'Y$, where $X$ is the matrix of independent variables and $Y$ is the column vector of observations.

The data consist of twelve observations of tool life at $f = .01725$ ipr for three speeds of 400, 500, and 600 sfpm. with a logarithmic transformation
of the cutting speeds, the matrix of independent variables \( \mathbf{X} \) is

\[
\begin{pmatrix}
X_0 \\
X_1 = \ln V
\end{pmatrix} =
\begin{pmatrix}
1 & 6.39693 \\
1 & 6.39693 \\
1 & 6.39693 \\
1 & 6.21461 \\
1 & 6.21461 \\
1 & 5.99146 \\
1 & 5.99146 \\
1 & 5.99146
\end{pmatrix}
\]

thus,

\[
(\mathbf{X}' \mathbf{X}) = \begin{pmatrix}
a_{00} & a_{01} \\
a_{10} & a_{11}
\end{pmatrix} = \begin{pmatrix}
12.00000 & 74.41200 \\
74.41200 & 461.75876
\end{pmatrix}
\]

and

\[
(\mathbf{X}' \mathbf{X})^{-1} = \begin{pmatrix}
a_{00} & a_{01} \\
a_{10} & a_{11}
\end{pmatrix} = \begin{pmatrix}
116.62377 & -18.79382 \\
-18.79382 & 3.03077
\end{pmatrix}
\]

The fitted equation is

\[
\hat{y} = \ln T = 35.66550 - 5.39915 \ln V
\]

i.e., \( b_0 = 35.66550 \) and \( b_1 = -5.39915 \). Therefore,

\[
n = -\frac{1}{b_1} = \frac{1}{5.39915} = .185
\]

\[
C = 2 \exp \left\{ -\frac{b_0}{b_1} \right\} = \exp \left\{ \frac{35.66550}{5.39915} \right\} = 740
\]

and the conventional form equivalent to equation (12) is

\[
VT^{.185} = 740
\]

This tool-life line is shown in Figure 1. The centroid (G) of the transformed
data is at $\bar{y} = 2.18550$ (i.e., $\bar{T} = 8.9$ minutes) and $\bar{x}_1 = 6.20100$ (i.e.,
$\bar{V} = 493$ sfpm).

The uncertainty of the least-squares estimates $b_0$ and $b_1$, or correspondingly the parameter estimates $n$ and $C$, is indicated by the 95% confidence intervals for $\beta_0$ and $\beta_1$, which represent plausible values for $b_0$ and $b_1$. However, it is necessary to take into account the high correlation between $b_0$ and $b_1$, as indicated by the magnitude of the covariance element $a_{01} = -18.79382$ in equation (11). That is, if the limits of the 95% confidence interval for $\beta_1$ are used as a measure of the uncertainty in estimating the slope $n$, then the proper limits of the 95% confidence interval for $\beta_0$ must be used to calculate the intercept $C$ by equation (9).

The confidence interval (CI) for a given parameter $\beta_1$, under the assumption of spherical normality, is

$$
CI (\beta_1) = b_1 \pm t_{v; \chi^2}^{2} (s_{\beta_1}^{2})^{1/2}
$$

(15)

where $t_{v; \chi^2}$ is Student's t-statistic with $v$ degrees of freedom and level of significance $\chi$, and where $a_{i1}^{11}$ is the element in the 1-th row and 1-th column of $(X'X)^{-1}$. The estimate of the error variance is

$$
s^2 = \frac{0.80972}{12-2} = 0.08097
$$

At $\gamma = 0.05$ and $v = 10$, $t_{10; 0.025} = 2.228$ and the 95% confidence interval for $\beta_0$ is

$$
CI (\beta_0) = 35.66550 \pm (2.228)(0.08097)(116.62377)^{1/2}
$$

$$
= \sqrt{42.51211}
$$

$$
\{28.81890
$$
and for $\beta_1$ is
\[
\text{CI}(\beta_1) = -5.39915 \pm (2.228 \{.08097 \ (3.03077)\}^{\frac{1}{2}}
= \{-4.29543
\begin{align*}
\text{limit of CI}(\beta_1) \text{ is used for } b_1, \text{ then the upper limit of CI}(\beta_0) \text{ is the proper value for } b_0. \text{ For instance, if } b_1 \text{ is set equal to } -6.50287, \text{ the lower 95% confidence limit for } \beta_1, \text{ the corresponding estimate for } b_0 \text{ is } 42.51211. \text{ Then, using the relationships given by equations (8) and (9)}
\]
\[n = \frac{1}{b_1} \quad \text{and } C = \exp\left(-\frac{b_0}{b_1}\right) = \exp\left(\frac{42.51211}{6.50287}\right) = 690
\]
\[n = -\frac{1}{6.50287} = .153
\]
\[C = \exp\left(-\frac{b_0}{b_1}\right) = \exp\left(\frac{42.51211}{6.50287}\right) = 690
\]
\[\text{and Taylor's equation is}
\]
\[v_T^{.153} = 690
\]
\[\text{Likewise, if } b_1 \text{ is set equal to the upper limit for } \beta_1, \text{ which is } -4.29543, \text{ the corresponding estimate for } b_0 \text{ is } 28.81890.
\]
\[n = \frac{1}{4.29543} = .233
\]
\[C = \exp\left(\frac{28.81870}{4.29543}\right) = 820
\]
\[\text{and hence Taylor's equation is}
\]
\[v_T^{.233} = 820
\]
\[\text{The two tool-life lines given by equations (17) and (19) are also shown in Figure 1. Both lines pass through the point (G) and when the slope}
\]
of the fitted lines is increased (e.g., \( n = 0.233 \)) the parameter \( C \) increases or if the slope is decreased (e.g., \( n = 0.153 \)) the parameter \( C \) also decreases. These lines indicate the limits of the fitted line which will reasonably approximate the true average tool life, whereas the tool-life line given by equation (14) is a prediction of the true average tool life based on the least-squares estimates for \( n \) and \( C \).

If the cutting speed is set at 567 sfpm, then based on equation (14) the predicted average tool life is 4.2 minutes as shown in Figure 1. The speed of 567 sfpm is the minimum-cost cutting speed for the numerical values given in Appendix B. If the 95% confidence values for \( n \) and \( C \) given in equations (16) and (18) are used to calculate \( V_{\text{min}} \), the limits of the \( V_{\text{min}} \) confidence interval are 535 and 628 sfpm respectively; and the corresponding limits of the predicted average tool life of 3.15 and 5.25 minutes are shown in Figure 1.

The unit-cost curves were also calculated using equation (1) to illustrate the effect of the uncertainty in \( n \) and \( C \) as shown in Figure 2. The minimum-cost cutting speeds are indicated by vertical lines from the minimum point of each unit-cost curve. Thus, \( V_{\text{min}} \) is not uniquely defined but lies within a relatively wide "95% confidence interval" of 93 sfpm, and a decision rule is needed in order to optimally choose a specific cutting speed from the \( V_{\text{min}} \) confidence interval.

**Minimax Decision Rule**

The selection of a particular cutting speed from the 95% confidence interval for \( V_{\text{min}} \) can be viewed in the following way. The decision is one
where the unit-cost curves for various possible combinations of $n$ and $C$ can be determined, but where the chance of a particular cost condition occurring is not known. Such decisions are grouped in a class called decisions under uncertainty, and there are several principles of choice which might be applied in order to make a decision under such circumstances. The conservative minimax principle is used to illustrate how a specific $V_{\text{min}}$ can be chosen from within its probable interval.

To apply the minimax principle in the determination of $V_{\text{min}}$ under uncertainty the maximum possible penalty-cost associated with each alternative speed within the 95% confidence interval is examined, and the cutting speed which minimizes the maximum penalty-cost is chosen. For instance, if in the previous example the operation was run at a cutting speed of 567 sfpm it would be expected that the minimum unit-cost would be $1.135 per workpiece as indicated by the middle unit-cost curve in Figure 2. (For convenience, this curve is labeled curve (b), the unit-cost curve based on $V_T = 690$ is labeled curve (a), and the curve based on $V_T = 820$ is labeled (c).) Now, if the correct unit-costs were instead given by curve (a), then the actual cost would be $1.155 per workpiece at $V = 567$ sfpm - resulting in a cost-penalty of 0.5 cent per workpiece from not operating at $V_{\text{min}} = 535$ sfpm for curve (a), where the minimum unit-cost would be $1.150 per workpiece. Likewise, if the unit-costs were instead given by curve(c), then there would be a cost penalty of 1 cent per workpiece from operating at 567 sfpm instead of 628 sfpm, the $V_{\text{min}}$ for curve (c).
sfpn. A more precise answer could be obtained if the number of alter-
natives to be considered is increased, which can easily be accomplished
by using a computer.

The Effect of the Cost-Time Parameter $t_e$

The cost-time parameter $t_e$ equals $t_c$ plus $c_t/c_o$; and as $t_e$ varies
because of changes in these cost and time parameters the unit-cost will
vary accordingly. Likewise, the minimum-cost cutting speed is an inverse
function of $t_e$ as shown by equation (2), and the derivation of $V_{\text{min}}$
depends on a correct determination of the cost and time parameters as noted in
previous studies (6, 8). The 95% confidence interval for $V_{\text{min}}$ is also
affected by the magnitude of $t_e$, which is illustrated by a continuation of
the previous example.

If $t_e$ is increased, say by an increase in the tool cost ($c_t$) or by an
increase in the tool-changing time ($t_c$), then: (i), the unit-cost at a
given cutting speed will be higher; (ii), the rate of change of the unit-
cost with cutting speed increases; (iii), the minimum-cost cutting speed
decreases; and (iv), the probable interval for \( V_{\text{min}} \) also decreases. On
the other hand, if \( t_e \) is increased by a decrease in the cost of operating
time \( (c_o) \), the minimum-cost cutting speed and its probable interval still
decrease, but the unit-cost and its rate of change also decrease.

Suppose the tool-changing time in the example is increased from
one-half to two minutes. The corresponding unit-cost curves and \( V_{\text{min}} \) ’s
for this new condition are shown in Figure 3. A comparison of Figure
2 \( (t_c = 0.5 \text{ and } t_e = 0.96) \) with Figure 3 \( (t_c = 2.0 \text{ and } t_e = 2.46) \) shows
that the unit-cost curves move up with the increase in \( t_e \), and that the minimum
unit-cost increases from approximately $1.15 to approximately $1.25 per workpiece.
This comparison also shows how the rate of change of the unit-cost in-
creases at the higher \( t_e \). Moreover, it can be seen that with this increase
in \( t_e \), \( V_{\text{min}} \) shifts to a lower speed range, and that the 95% confidence in-
terval for \( V_{\text{min}} \) is reduced from 93 to 41 sfpm, i.e., the probable interval
for \( V_{\text{min}} \) changes from 535-628 sfpm to 463-504 sfpm.

If for this higher tool-changing time of two minutes the cost of operat-
ing time \( (c_o) \) is also reduced from $0.22 to $0.10 per minute, then \( t_e =
3.0 \) and the corresponding unit-cost curves and \( V_{\text{min}} \) ’s are shown in Figure
4. With this larger \( t_e \), \( V_{\text{min}} \) shifts to an even lower speed range and the
95% confidence interval for \( V_{\text{min}} \) is now 449-481 sfpm, a range of only
32 sfpm. However, the unit-cost at a given speed and the rate of change
of the unit-cost decreases, as shown by a comparison of Figures 3 and 4.

The effect of \( t_e \) on the 95% confidence interval for \( V_{\text{min}} \) can be further illustrated by means of the theoretical tool-life for minimum cost \( (T_{\text{min}}) \). This minimum-cost tool life has, for a given \( t_e \), a confidence interval analogous to that for \( V_{\text{min}} \). This is because \( T_{\text{min}} \) is also a function of the slope \( n \), i.e.,

\[
T_{\text{min}} = \left( \frac{1}{n} - 1 \right) t_e
\]  

(20)

Hence, if \( n \) varies from .153 to .233, then under the original conditions where \( t_e = 0.96 \) minutes \( T_{\text{min}} \) varies from 5.25 to 3.50 minutes; whereas if \( t_e = 2.46 \) minutes \( T_{\text{min}} \) ranges from 13.5 to 8.1 minutes. These confidence intervals for \( T_{\text{min}} \) with the corresponding 95% confidence intervals for \( V_{\text{min}} \) are shown in Figure 1.

As \( t_e \) increases the \( T_{\text{min}} \) confidence interval increases, e.g., from 2.1 minutes for \( t_e = 0.96 \) to 5.4 minutes for \( t_e = 2.46 \); but the corresponding \( V_{\text{min}} \) confidence interval decreases from 93 to 41 sfpm. The confidence interval for \( T_{\text{min}} \) at \( t_e = 3.00 \), which is not shown in Figure 1, has a range of 6.6 minutes and the 95% confidence interval for \( V_{\text{min}} \) decreases to only 32 sfpm as noted. Thus, as the magnitude of the cost-time parameter \( t_e \) increases the \( V_{\text{min}} \) confidence interval decreases.

The Effect of Feed

The importance of including feed as a variable in the determination of the optimum cutting conditions has been shown in prior studies (1, 6, 8). If feed is incorporated into the unit-cost model with Taylor's expanded
tool-life equation \( V_t^n f^m = k \), the minimum-cost cutting speed is given by

\[
V_{\text{min}} = \frac{k}{f^m \left[ \left( \frac{1}{n} - 1 \right) t_e \right]^n}
\]

(21)

Hence, \( V_{\text{min}} \) is not only affected by the uncertainty in the estimation of \( n \) and \( k \), but also in the estimation of \( m \) as well.

To illustrate the effect of experimental error on \( V_{\text{min}} \) when feed is included as a variable, the complete set of carbide tool-like data in Appendix A is used. These data consist of 36 observations of tool-like taken over a cutting-speed range of 400-800 sfpm and at three different levels of feed—0.157, .01725, and .022 1pr. With a logarithmic transformation of the experimental cutting conditions the \((X'X)\) matrix is

\[
(X'X) = \begin{bmatrix}
    a_{00} & a_{01} & a_{02} \\
    a_{10} & a_{11} & a_{12} \\
    a_{20} & a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix}
    36.00000 & 228.44224 & -144.96532 \\
    1451.28808 & -920.10261 & 584.25315
\end{bmatrix}
\]

and

\[
(X'X)^{-1} = \begin{bmatrix}
    a_{00} & a_{01} & a_{02} \\
    a_{10} & a_{11} & a_{12} \\
    a_{20} & a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix}
    45.91874 & -2.93726 & 6.76769 \\
    0.62690 & 0.25847 & 2.08797
\end{bmatrix}
\]

The postulated model to be fitted is now

\[
E(y) = E(\ln T) = \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2
\]

(22)

where \( E(y) \), \( \beta \), \( x_0 \), and \( x_1 \) have been defined and where \( x_2 \) is the feed on a logarithmic scale. Fitting equation (22) by least squares gives...
\[ \hat{y} = \ln T = 18.74963 - 5.12088 \ln V - 3.73786 \ln f \]

i.e., \( b_0 = 18.74963 \), \( b_1 = -5.12088 \), and \( b_2 = -3.73786 \); and the estimate of the error variance is

\[ s^2 = \frac{1.82098}{33} = .055 \]

A measure of the uncertainty in the "b" 's is given by their confidence intervals, which are:

95% CI(\( \hat{b}_0 \)) = 18.74963 ± (2.036) \( \left[ (45.91874) (.055) \right]^{1/2} \]

\[ \begin{align*}
    &\approx [21.99053, 15.50877]
\end{align*} \]

95% CI(\( \hat{b}_1 \)) = -5.12088 ± (2.036) \( \left[ (45.62690) (.055) \right]^{1/2} \]

\[ \begin{align*}
    &\approx [-5.49955, -4.74219]
\end{align*} \]

95% CI(\( \hat{b}_2 \)) = -3.73786 ± (2.036) \( \left[ (20.08797) (.055) \right]^{1/2} \]

\[ \begin{align*}
    &\approx [-4.42784, -3.04788]
\end{align*} \]

However, the estimates \( b_0 \), \( b_1 \), and \( b_2 \) are, as in the two parameter case, correlated with each other. Therefore, the individual confidence limits are inadequate for interpreting the joint uncertainty in the estimation of \( b_0 \), \( b_1 \), and \( b_2 \).

To interpret this joint uncertainty, \( b_0 \) and \( b_1 \) can be estimated independently of \( b_2 \) by an orthogonal transformation of the variables \( x_0 \) and \( x_1 \).

This orthogonality is induced by rewriting equation (22) as:

\[ E(y) = E(\ln T) \]

\[ = \beta_0 (x_0 - A_2 x_2) + \beta_1 (x_1 - A_2 x_2) + (\beta_2 A_2 + A_2 \beta_0 + A_2 \beta_1) x_2 \]

i.e.,

\[ E(y) = E(\ln T) = \beta_0 x_{o2} + \beta_1 x_{12} + \beta_2 x_2 \]

(23)
where

\[
\begin{align*}
x_{O,2} &= x_O - A_{O,2} x_2 \\
x_{1,2} &= x_1 - A_{1,2} x_2 \\
\beta_2' &= \beta_2 + A_{O,2} \beta + A_{1,2} \beta_1,
\end{align*}
\]

and where \( A_{O,2} = (x_2' x_2)^{-1} x_2' x_O \) and \( A_{1,2} = (x_2' x_2)^{-1} x_2' x_1 \).

The general form of the corresponding predicting equation is

\[
\hat{y} = \ln T = b_0 x_{O,2} + b_1 x_{1,2} + b_2' x_2
\]

(24)

With this orthogonal transformation of \( x_O \) and \( x_1 \), the \((X'X)\) matrix

is:

\[
(X'X) = \begin{pmatrix}
0.03110 & 0.14570 & 0 \\
0.14570 & 2.77782 & 0 \\
0 & 0 & 584.25315
\end{pmatrix}
\]

and its inverse is:

\[
(X'X)^{-1} = \begin{pmatrix}
45.91873 & -2.93726 & 0 \\
-2.93726 & 0.62690 & 0 \\
0 & 0 & 0.00171
\end{pmatrix}
\]

Therefore, the estimation of \( b_0 \) and \( b_1 \) is independent of the estimation of \( b_2' \), and the effect of experimental error for a given feed can be carried out as in the example.

Fitting equation (24) by least squares to the 36 observations on a logarithmic scale gives

\[
\hat{y} = \ln T = 18.74963 x_{O,2} - 5.12088 x_{1,2} - 0.32548 x_2
\]

(25)

Under the orthogonalizing transformation the coefficients \( b_0 \) and \( b_1 \) remain the same, and the only new coefficient is \( b_2' = -0.32548 \). Also under
such a transformation the estimate of the error variance is unchanged, \( S^2 = 0.055 \), and the 95% confidence intervals for \( \beta_0 \) and \( \beta_1 \) are the same as before.

In order to convert from \( b_0 \) and \( b_1 \) in the linear model given by equation (24) to the estimate \( C \) for a given feed a new transforming relationship is needed. This new relationship can be determined by noting that for a given feed equation (24) can be written as

\[
\hat{y} = \ln T = \left[ b_0 (x_0 - A_{o2} x_2) - A_{12} x_2 b_1 + b_2' x_2 \right] + b_1 x_1
\]

Comparing equations (7) and (26) \( n \) is still given by equation (8), but \( C \) is now given by:

\[
C = \exp \left\{ - \frac{b_0 (x_0 - A_{o2} x_2) - A_{12} x_2 b_1 + b_2' x_2}{b_1} \right\}
\]

Then, for example, at \( f = 0.022 \) ipr and using equations (8) and (27), equation (25) can be written as:

\[
V_T^{195} = 631
\]

Using the appropriate limits of the 95% confidence intervals for \( \beta_0 \) and \( \beta_1 \), the corresponding Taylor's equations are:

\[
V_T^{182} = 632
\]

and

\[
V_T^{211} = 639
\]

Based on the two tool-life equations given by equations (29) and (30) and the same numerical values given in the example (with \( t_c = 2 \) minutes) the 95% confidence interval for \( V_{\min} \) is 395-408 sfpm, a range of 13 sfpm.

Assuming that the tool-life equation developed from the 36 observations is valid over a cutting speed range of 200-1,000 sfpm and a feed range of
.0075 - .030 ipr, the three Taylor's equations and corresponding minimum-cost cutting speeds at various feeds can likewise be determined and are shown in Table 2, which summarizes the change in the width of the 95% confidence interval for $V_{\text{min}}$.

From Table 2, it can be seen that the 95% confidence interval for $V_{\text{min}}$ increases as the feed changes above or below $f = .01725$ ipr. This trend occurs because $f = .01725$ ipr is very close to the average experimental feed of .018 ipr for the 36 observations. This average feed is a coordinate of the centroid $(\bar{X}_1, \bar{X}_2, \bar{y})$ of the fitted tool-life plane, and the uncertainty from the experimental error has the smallest effect at the centroid, which accounts for the trend of $V_{\text{min}}$. Thus, the experimental range of cutting conditions in the tool-life tests affects the probable interval for $V_{\text{min}}$.

The five corresponding families of unit-cost curves at each feed are shown in Figure 5. The minimum-cost cutting speeds are indicated by the vertical lines from the minimum of each unit-cost curve, and it can be seen how the confidence interval for $V_{\text{min}}$ increases for feeds above and below $f = .001725$ ipr. These curves also show that the minimum unit-cost decreases as the feed increases. In addition, they illustrate how the rate-of-change of the unit-cost with cutting speed varies with feed, and that the unit-cost is more sensitive to deviations from the minimum-cost cutting speed at higher feeds. These latter results are consistent with those noted in previous analyses (1, 8) and are true in general.
CONCLUSIONS

1. The effect of experimental error on the determination of the minimum-cost cutting speed was investigated using the concept of statistical inference.

2. The theoretical cutting speed for minimum cost is not uniquely defined, but lies within a probable range of speeds because of the uncertainty in the parameter-estimates of the tool-life equation.

3. The confidence interval for $V_{\text{min}}$ depends on the magnitude of the cost-time parameter $t_e$; as $t_e$ increases the $V_{\text{min}}$ confidence interval decreases.

4. The confidence interval for $V_{\text{min}}$ is affected by the experimental range of feed in tool-life testing; the smallest confidence interval is at the average experimental feed.

5. A decision rule based on the minimax principle is used to illustrate the selection of a specific speed from within the $V_{\text{min}}$ confidence interval.
REFERENCES


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* The tool-life equations given by (b) are based on the least squares estimates for $b_0$ and $b_1$. The tool-life equations given by (a) and (c) are based on the appropriate limits of the 95% confidence intervals for $b_0$ and $b_1$. 
APPENDIX A

Tool-life data. (Tool material - carbide, work material - SAE 4340; BHN 200, depth of cut - .100 in.) (δ)

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<th>f-ipr</th>
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APPENDIX B

The following numerical values were used to calculate the unit cost and the minimum-cost cutting speed in the example (3):

Workpiece diameter is 6 inches
Length of cut is 16 inches
Depth of cut is 0.100 inches

c_t is $.10 per cutting edge
t_c is 0.5 minute per edge

c_o is $.22 per minute
t_h is 2.0 minute per edge
CAPTIONS FOR THE TABLES AND ILLUSTRATIONS

Table 1  Minimax Decision Matrix.

Table 2  The effect of experimental error on the minimum-cost cutting speed at various feeds.

Figure 1  The effect of experimental error on the tool-life line, $V_{\text{min}}$, and $T_{\text{min}}$.

Figure 2  The effect of experimental error on the unit-cost and the minimum-cost cutting speed. ($t_e = 0.96 \text{ min/edge}$)

Figure 3  The effect of experimental error on the unit-cost and the minimum-cost cutting speed. ($t_e = 2.46 \text{ min/edge}$)

Figure 4  The effect of experimental error on the unit-cost and the minimum-cost cutting speed. ($t_e = 3.00 \text{ min/edge}$)

Figure 5  The effect of experimental error on the unit-cost and the minimum-cost cutting speed at various feeds.
Cutting Speed-Tool Life

\[ f = 0.01725 \text{ ipr} \quad d = 0.100 \text{ in} \]

SAE 4340 (200 BHN)

12 Observations

Cutting Speed \( V_T \), sfpm

Tool Life \( T \), min.
$c_o = \$0.22/\text{min}$  
$c_t = \$0.10/\text{edge}$

$t_c = 0.5 \text{ min}$  
$t_h = 2.0 \text{ min}$

$f = 0.01725 \text{ ipr}$  
$d = 0.100 \text{ in}$

$L = 16.0 \text{ in}$  
$D = 6.0 \text{ in}$

$V_{T.153} = 690$

$V_{T.185} = 740$

$V_{T.233} = 820$

Unit Cost $C_u, \$/pc

Cutting Speed $V, \text{ sfpm}$

Curve (a)

Curve (b)

Curve (c)
$c_o = \$0.22/\text{min} \quad c_t = \$0.10/\text{edge}$

$\tau_c = 2.0 \text{ min} \quad \tau_h = 2.0 \text{ min}$

$f = 0.01725 \text{ ipr} \quad d = 0.100 \text{ in}$

$L = 16.0 \text{ in} \quad D = 6.0 \text{ in}$
Unit Cost $C_u$, $$/pc.$

- $c_o = $.10/min
- $c_t = $.10/edge$
- $t_c = 0.5 \text{ min}$
- $t_h = 2.0 \text{ min}$
- $f = .01725 \text{ ipr}$
- $d = 0.100 \text{ in}$
- $L = 16.0 \text{ in}$
- $D = 6.0 \text{ in}$

Cutting Speed $V$, sfpm

$VT_{153} \approx 690$
$VT_{185} \approx 740$
$VT_{233} \approx 820$

FIGURE 4
\[ c_0 = \$0.22/\text{min} \quad c_t = \$0.10/\text{edge} \]
\[ t_c = 2.0 \text{ min} \quad t_h = 2.0 \text{ min} \]
\[ f = 0.01725 \text{ ipr} \quad d = 0.100 \text{ in} \]
\[ L = 16.0 \text{ in} \quad D = 6.0 \text{ in} \]