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An Exploratory Study of
Taylor's Tool-Life Equation By
Power Transformations

by

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Abstract

Transformations of both dependent and independent variables are employed to investigate the linearization of Taylor's tool-life equation. This exploratory study indicates that a logarithmic transformation, which is a special case of the general class of power transformations, gives the best fit for HSS tool-life data. However, the study does not show that a logarithmic transformation is the best for carbide tool-life data. For a wide cutting range, where Taylor's tool-life equation does not hold, a linear equation instead of a second-order relationship for the prediction of tool life can be determined by the proper transformation.

Introduction

It has long been recognized that over a relatively narrow cutting-speed range, a straight line satisfactorily fits experimental tool-life data plotted on a $\ln - \ln$ scale. This linear relationship is known as Taylor's tool-life equation:

$$VT^n = C \quad (1)$$

where $V$ is the cutting speed in sfpm, $T$ is the tool life in minutes, $n$ is a constant equal to the slope of the straight line, and $C$ is a constant equal to the cutting-speed for one minute of tool life. An extension of this equation is the generalized tool-life equation:

$$VT^{n_f m_d} = K \quad (2)$$
where \( f \) is the feed in inches per revolution, \( d \) is the depth of cut in inches, and \( m, p \) and \( k \) are constants.

The advantage of fitting tool-life data by Taylor's logarithmic transformation is the linearization of the tool-life equation, with the result that the meaning of the predicting equation can be more easily understood. However, it has never been determined if this linear equation, defined by a logarithmic transformation, yields the best fit for tool-life data. It is therefore possible that some other transformation, such as a reciprocal or square-root transformation, could be used to linearize the tool-life equation and give a better fit.

Furthermore, Taylor's tool-life equation is only valid within a relatively narrow cutting-speed range, and a linear predicting equation which fits the tool-life data for a wide cutting range is not available. Second-order tool-life relationships that give an adequate fit over wide ranges have been investigated \([1,2]\) \(^1\), but the meaning of these second-order relationships is less easily understood. Therefore, it is desirable that the proper transformation be determined such that a linear equation in the transformed variables can be used to fit the tool-life data over a wide range of cutting conditions.

The purpose of this paper is to search for the best transformation

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\(^1\) Numbers in brackets designate references at the end of the paper.
for tool-life data, and to determine a linear equation which can adequately predict tool-life over a wide operating range. The residual sum of squares (RSS) calculated by the least squares method is used as the numerical criterion to indicate the best fit. The theory of transformations is briefly explained in Appendix A.

The transformation of equation (2) involves transformations of both the dependent variable \( T \) and the independent variables \( V, f, \) and \( d \). The transformations used in this study for the dependent variable are of the form \([5]\):

\[
T(\lambda) = \begin{cases} \frac{T^\lambda - 1}{\lambda (T)^{\lambda - 1}} & \lambda \neq 0 \\ \ln T & \lambda = 0 \end{cases}
\]

where \( \bar{T} \) is the geometrical mean of the observations, i.e.,

\[ \bar{T} = \left( \frac{T_1 \times T_2 \times \ldots \times T_i \times \ldots \times T_n}{n} \right)^{\frac{1}{n}} \]

where \( n \) is the number of observations and \( T_i \) is the \( i \)-th observed tool life.

For the independent variables the transformations are of the form \([4]\):

\[
U_i = \begin{cases} x_i^{\alpha_i} & \alpha_i \neq 0 \\ \ln x_i & \alpha_i = 0 \end{cases}
\]

Combining these transformation of the dependent and independent variables, the generalized tool-life relationship is of the form:

\[
E \left[ T(\lambda) \right] = \beta_0 + \beta_1 U_1 + \beta_2 U_2 + \beta_3 U_3
\]

i.e.,

\[
E \left[ T(\lambda) \right] = \beta_0 + \beta_1 V^{\alpha_1} + \beta_2 f^{\alpha_2} + \beta_3 d^{\alpha_3}
\]
where $E[T^{(\lambda)}]$ is the expected value of the transformed tool life; $\lambda$, $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the transforming parameters; and $\beta_0$, $\beta_1$, $\beta_2$ and $\beta_3$ are the parameters for the transformed tool-life equation.

When $\lambda = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$, equation (5) can be written as:

$$E \left[ T \ln T \right] = \beta_0 + \beta_1 \ln V + \beta_2 \ln f + \beta_3 \ln d \quad (6)$$

Except for the constant $\hat{T}$, equation (6) is the familiar logarithmic transformation of the generalized tool-life equation $VT^n f^m d^p = K$, and thus a logarithmic transformation is only a special case of the transformations discussed in this study.

If the depth of cut is constant, equation (5) reduces to the form:

$$E \left[ T^{(\lambda)} \right] = \beta_0 + \beta_1 V^{\alpha_1} + \beta_2 f^{\alpha_2} \quad (7)$$

which is used to fit the data in all three parts of the following study.

The first part is an analysis of the best transformation for high-speed steel tool-life data, and the second part is a similar analysis for carbide data. In the third part of this study various transformations are analyzed in order to determine a linear equation which satisfactorily fits carbide tool-life data taken over a wide operating range.

**High-Speed Steel (HSS) Tool-Life Data**

This part of the study is based on a set of 50 observations of tool life obtained from cutting SAE 1045 with a HSS tool, with feeds ranging
from .0007 to .0228 in, with cutting speeds ranging from 80 to 548 sfpm, and a constant depth of cut of .050 inches. The data were taken at the University of Michigan in the spring of 1964 and furnished by Professor J. Datsko.

Plotting a representative subset of these tool-life data on three different sets of scales: (a) In $V$ vs In $T$, (b) $V$ vs In $T$ and (c) $V^{1/2}$ vs In $T$, a family of straight lines can be fitted to the data as shown in Figures 1(a), 1(b) and 1(c) respectively. It can be seen that in all three cases the straight lines appear to fit the data satisfactorily. That is, it cannot be detected visually which are the best scales for plotting the tool-life data, or equivalently, which are the best transformations for fitting the data. However, a quantitative comparison can be made by calculating the RSS for each transformation, when fitting the tool-life equations by the method of least squares. The resulting residual sums of squares shown in Table I, where the tool-life equations are based on the complete set of 50 observations and assuming a logarithmic transformation of $f$, indicate that a logarithmic transformation of all the variables yields the best fit among the three. But it should be noted that these transformations of the dependent and independent variables are just special cases of the general class of power transformations, and are not necessarily the best transformations since there are many other combinations of power transformations which could be used to linearize
TABLE 1

<table>
<thead>
<tr>
<th>Tool-Life Equation</th>
<th>RSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln \hat{T} = 30.33 - 10.25 \ln V - 4.64 \ln f )</td>
<td>28.53</td>
</tr>
<tr>
<td>( \ln \hat{T} = -2.76 - 1.09 V^{1/2} - 3.65 \ln f )</td>
<td>41.57</td>
</tr>
<tr>
<td>( \ln \hat{T} = -5.31 - 0.02 V - 1.97 \ln f )</td>
<td>60.45</td>
</tr>
</tbody>
</table>

the predicting equation.

In order to search for the best transformation, the RSS for many possible combinations of \( \lambda_1 \), \( \alpha_1 \) and \( \alpha_2 \) can be determined with the aid of a high speed computer. To simplify the analysis \( \alpha_2 \) is tentatively held constant, and the RSS is calculated for various combinations of \( \lambda \) and \( \alpha_1 \) as described in Appendix B. For instance, when \( \alpha_2 = 0 \) (i.e., assuming a logarithmic transformation of feed), \( \alpha_1 = 0.2 \) and \( \lambda = -0.4 \), the RSS resulting from fitting equation (7) to the data is 144 as shown at the point \( (\lambda = -0.4, \alpha_1 = 0.2) \) on the RSS grid in Figure 2. If the same RSS values on this grid in the \( \lambda-\alpha_1 \) plane are connected together, contour lines of constant RSS can be drawn on the grid. A representative group of RSS contours of 250, 200, 150, 124, and 112 are shown in Figure 2.

From such a contour diagram the relative effect on the RSS from
varying \( \lambda \) and \( \alpha_1 \) can be visually observed. The region where the minimum RSS occurs can also be seen, and hence, the transformation \((\hat{\lambda}, \hat{\alpha}_1)\) which gives the best fit can be determined. In addition, a 95% confidence region (i.e., with 95% confidence it can be said that the true values of the transforming parameters \( \lambda \) and \( \alpha_1 \) lie within this region) can be determined as described in Appendix A. This region is shown by the cross-hatched area in Figure 2 and indicates the most likely values of \( \lambda \) and \( \alpha_1 \), whereas \( \hat{\lambda} \) and \( \hat{\alpha}_1 \) are the best point estimates of the transforming parameters.

It is noteworthy that for this HSS tool-life data the RSS of 113 at the point \((\lambda = 0, \alpha_1 = 0)\) is well within this 95% confidence region, and in fact, is approximately the minimum RSS. In other words, an equation of the form

\[
E[\hat{T} \ln T] = \beta_0 + \beta_1 \ln V + \beta_2 \ln f
\]

(8)

almost yields the best fit.

The RSS values shown on the contour diagram are standardized values which permit the comparison of the RSS over all values of \( \lambda \). The standardized RSS (113) at the particular point \((\lambda = 0, \alpha_1 = 0)\) can be converted to the RSS scale shown in Table 1 by dividing the standardized RSS by the square of the geometrical mean \( \hat{T} \), which is 2.0 for this initial set of data. That is, the RSS in Table I for the logarithmic transformation is \( \frac{113}{4} = 28.5 \).
To further investigate the possibility that some transformation other than that indicated above may produce a better fit, various \( \alpha_2 \)'s were studied. For instance, a RSS contour diagram for \( \alpha_2 = 1 \) (i.e., no transformation of \( f \)) is shown in Figure 3, where it can be seen that the minimum RSS (242) is much larger than the minimum RSS (110) in Figure 2 where \( \alpha_2 = 0 \). Furthermore, the 95% confidence region in Figure 3 (only a little over half the contour diagram is shown) is much larger than that in Figure 2. From this comparison it is obvious that \( \alpha_2 = 0 \) is a better transformation than \( \alpha_2 = 1 \). Such comparisons for various \( \alpha_2 \)'s showed that \( \alpha_2 = 0 \) is the best transformation. Therefore, based upon one set of data, this exploratory study does indicate that Taylor's logarithmic transformation is the best for HSS tool-life data.

*Carbide Tool-Life Data*

Since the invention of the cemented carbide, Taylor's logarithmic transformation has also been used to fit carbide tool-life data. From previous investigations [1] it appears that a logarithmic transformation of both the dependent and independent variables is adequate. It is not necessarily true, however, that this logarithmic transformation is also the best for carbide tool-life data, even though it is the best for HSS.

The carbide tool-life data used for this part of the study is taken from a previously published government report [3], and include two separate sets of data taken at a constant depth of cut of 0.100 inches.
One set of 36 observations was obtained by cutting SAE 4340 at 200 BHN with a C-6 carbide tool, with feeds ranging from .0157 to .022 ipr, and with cutting-speeds ranging from 400 to 800 sfpm (Appendix B Table B1). The other set of 30 observations is from cutting SAE 4340 at 300 BHN with a C-7 carbide tool, with feeds ranging from .011 to .022 ipr, and with cutting speeds ranging from 600 to 900 sfpm.

Using a subset of 20 observations taken at a constant feed \( f = .01725 \text{ ipr} \), from the first set of 36 observations a straight line appears to satisfactorily fit these observations plotted on three different sets of scales: (a) \( \ln V \) vs \( \ln T \), (b) \( V \) vs \( \ln T \), and (c) \( V \) vs \( T^{(-2)} \), as shown in Figures 4(a), (b), and (c) respectively. As with the HSS data it cannot be detected visually which transformations yield the best fit. However, when a linear relationship in the transformed variables is fitted by least squares to the complete set of 36 observations, the resulting standardized residual sums of squares shown in Table 2 indicate that the best fit among the three predicting equations is obtained by a transformation employing \( \lambda = -0.2 \), \( \alpha_1 = 1.0 \) and \( \alpha_2 = 0 \).
TABLE 2

<table>
<thead>
<tr>
<th>Tool-Life Equation</th>
<th>RSS (standardized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{(-2)} = -29.42 - 0.32 V - 13.24 \ln f$</td>
<td>20.99</td>
</tr>
<tr>
<td>$T \ln T = 69.22 - 18.9 \ln V - 13.8 \ln f$</td>
<td>24.82</td>
</tr>
<tr>
<td>$T \ln T = -33.16 - .03 V - 14.16 \ln f$</td>
<td>31.25</td>
</tr>
</tbody>
</table>

Since these predicting equations are the result of particular transformations defined by equations (3) and (4), it is necessary to investigate other combination of $\lambda$, $\alpha_1$, and $\alpha_2$ in order to determine in general which combination gives the minimum RSS (or best fit) for this set of carbide data. The effect on the RSS for many possible combinations of $\lambda$, $\alpha_1$, and $\alpha_2$ can again be more readily seen by constructing RSS contour diagrams as described in the previous section.

RSS contour diagrams in the $\lambda-\alpha_1$ plane for $\alpha_2 = 0$ and $\alpha_2 = 1$ were constructed for each group of data and are shown in Figures 5 to 8. The RSS contour diagrams in Figure 5 ($\alpha_2 = 0$) and Figure 6 ($\alpha_2 = 1$) are based on the first set of 36 observations, and the RSS contour diagrams in Figure 7 ($\alpha_2 = 0$) and Figure 8 ($\alpha_2 = 1$) are based on the second set of 30 observations. (It should be noted that the horizontal scale in Figures 7 and 8 is different than that in Figures 5 and 6.)
A comparison of these RSS contour diagrams within both pairs of Figures (5 vs 6 and 7 vs 8) shows that the shape of the contours is the same, that the minimum RSS occurs at approximately the same point, and that the minimum value is not much different. These similarities disclose that the transformation of $f$ is unimportant in determining the best transformations for $T$ and $V$. This disclosure is in sharp contrast with the previous HSS analysis where the RSS contour diagrams for $\alpha_2 = 0$ and $\alpha_2 = 1$ were completely different, indicating that the proper transformation of $f$ was important for HSS tool-life data.

For the first group of 36 observations the minimum RSS occurs at approximately the point $\hat{\lambda} = -0.25$ and $\hat{\alpha}_1 = 0.75$ on the RSS contour diagram in Figure 5. Therefore, the linear relationship in the transformed variables which fits the data best is of the form:

$$E \left[ \frac{T^{-.25} - 1.0}{(-.25)(\hat{T})^{-1.25}} \right] = \beta_0 + \beta_1 V^{0.75} + \beta_2 \ln f.$$  \hspace{1cm} (9)

The transformation $\alpha_2 = 1$ could also be used since the minimum residual sums of squares in Figures 5 ($\alpha_2 = 0$) and 6 ($\alpha_2 = 1$) are not of significant difference. It can also be seen from these diagrams that the point $\lambda = 0$ and $\alpha_1 = 0$ (RSS = 25) lies just outside the 95% confidence regions indicating that a logarithmic transformation is probably inadequate.

This inadequacy of the logarithmic transformation is also true for the second set of 30 observations since the point $\lambda = 0$ and $\alpha_1 = 0$ (RSS = 27) likewise lies outside the 95% confidence regions shown in
Figures 7 and 8. However, the minimum RSS now occurs at approximately
the point $\hat{\lambda} = 0.1$ and $\hat{\alpha}_1 = 2.8$ in Figure 8, and therefore, the linear
relationship in the transformed variables which fits the 30 observations
best is of the form:

$$E \left( \frac{T^{0.1} - 1.0}{(0.1) \ (\bar{T})^{-0.9}} \right) = \beta_0 + \beta_1 V^{2.8} + \beta_2 f. \tag{10}$$

Although, the transformation $\alpha_2 = 0$ (i.e. a logarithmic transformation
of $f$) could also be used since the minimum residual sums of squares
in Figures 7 and 8 are also not significantly different.

Thus, an analysis of these two sets of carbide tool-life data,
by means of RSS contour diagrams, indicates that a logarithmic transfor-
mation of both the dependent variable (tool-life) and the independent
variables (speed and feed) does not give the best fit. However, in order
to determine the best transformations to linearize carbide tool-life data,
a more comprehensive analysis of experimental carbide data is needed.
It is hoped that this exploratory study will create an interest in searching
for the proper transformation to determine a linear predicting equation
which will best fit carbide tool-life data.

The Prediction of Tool Life Over a Wide Operating Range

It is well-known that beyond a certain operating range the linear
tool-life equation defined by a logarithmic transformation does not satis-
factorily predict tool life. In order to obtain a better predicting relationship
over a wide range of operating conditions more comprehensive second-order tool-life equations have been investigated by Colding [2] and Wu [1]. Such second-order equations give a better fit than a linear equation, but lack the advantage of simplicity which is important in engineering applications.

This part of the analysis is based on tool-life data taken over a wide cutting-speed range (226 - 1020 sfpm). These data consist of 24 observations when cutting SAE 1018 with a carbide tool (Carboloy 162P4, 78B), with feeds ranging from .00725-.034 ipr, and depths of cut ranging from .034 - .143 inches. To simplify the analysis it was assumed that the observations were taken at a constant depth of cut, and hence equation (7) was used.

To investigate the relative effect on the RSS when fitting equation (7) under various possible transformations, RSS contour diagrams and 95% confidence regions were determined for two transformations of f and are shown in Figures 9 ($\alpha_2 = 0$) and 10 ($\alpha_2 = 1$). Since the minimum RSS of 2950 occurs at approximately the point ($\tilde{\lambda} = 0.3$, $\tilde{\alpha}_1 = 0.1$) on the contour diagram in Figure 10, the linear relationship which fits the observations best is of the form:

$$E \left[ \frac{T^{0.3} - 1.0}{(0.3) (T)^{-0.7}} \right] = \beta_0 + \beta_1 V^{0.1} + \beta_2 f$$

(11)

However, the transformations shown in Figure 9 also define a linear equation at a minimum RSS of 3100. This linear equation in the transformed variables is of the form

$$E \left[ \frac{T^{0.3} - 1.0}{(0.3) (T)^{-0.7}} \right] = \beta_0 + \beta_2 \ln V + \beta_3 \ln f$$

(12)

which can also be considered as a good transformation due to the simplicity and ease of interpretation.
To demonstrate that the proper transformation can define a linear equation which fits experimental tool-life data taken over a wide range of cutting speed, a series of straight lines was fitted to a subset of the 24 observations plotted on four different sets of scales as shown in Figures 11 (a), (b), (c) and (d). This subset consists of 8 observations taken at a constant feed (.015 ipr), a constant depth of cut (.070 inch), and with cutting speeds ranging from 226 to 1020 sfpm.

In Figure 11 (a) the 8 observations are plotted on logarithmic scales, and it is obvious that a straight line does not fit the data. Figure 11(b) shows that a straight line also does not fit the observations when plotted on a semi-logarithmic scale (i.e., V vs ln T). This lack of fit shown in both 11(a) and (b) is also indicated by the location of the RSS for the corresponding transformations on the contour diagrams in both Figures 9 and 10, where the point ($\alpha_1 = 0$, $\lambda = 0$), corresponding to the logarithmic scaling in 11(a), and the point ($\alpha_1 = 1$, $\lambda = 0$), corresponding to the semi-logarithmic scaling in 11(b), both lie outside the 95% confidence region. Therefore, it is unlikely that logarithmic transformations will be satisfactory for predicting tool life over a wide cutting-speed range, which is in agreement with what has long been recognized.

If the transformations are based on an analysis of the RSS contour diagram shown in Figure 9, the minimum RSS is obtained for the transformations $\hat{\alpha}_1 = 0$ and $\hat{\lambda} = .3$. Plotting the 8 observations on scales corresponding to these transformations (i.e., ln V vs $T^{(.3)}$), as in Figure 11(c), it can clearly be seen that a straight line fits the observations over the complete cutting-speed range. Therefore, an optimum transformation to linearize the tool-life data over a wide operating range can be determined.
Although the minimum RSS in Figure 9 occurs at the point $(\hat{\alpha}_1 = 0, \hat{\lambda} = .3)$, any combination of $\alpha_1$ and $\lambda$ within the 95% confidence region can yield a reasonable transformation. For example, if we choose the combination $\alpha_1 = .5$ and $\lambda = 0$, which is well within the 95% confidence regions in both Figures 9 and 10, the 8 observations can be plotted on the corresponding scales (i.e., $V^\frac{1}{2}$ vs $\ln T$), as in Figure 11(d), to show that a straight line still gives a good fit. The RSS at this point ($\alpha_1 = .5, \lambda = 0$) in Figure 10 is approximately 3200, and any other combination of $\lambda$ and $\alpha_1$ (e.g., $\alpha_1 = 0, \lambda = .5$) which are denoted by this contour will also define a linear relationship which gives a good fit. Therefore, there is no unique combination of transformations which give a satisfactory fit, and the transformations can be selected taking into account considerations of simplicity and ease of interpretation.

Thus, this part of the exploratory study shows that over a wide operating range a linear equation, instead of a second-order equation, for the prediction of tool life can be determined by proper transformations of both the dependent and independent variables.

Conclusions

From the exploratory study of the linearization of the tool-life equation it is tentatively concluded that:

1. The application of transformation theory indicates that a logarithmic transformation of the tool-life equation best fits HSS data.

2. Although more carbide tool-life data needs to be analyzed in order to determine the best transformation, a limited analysis suggests that a logarithmic transformation is not the best for carbide data.
3. A linear equation for the prediction of tool-life over a wide operating range can be determined by proper transformations.

Acknowledgment

The authors wish to thank Professor J. Datsko of the University of Michigan for furnishing his unpublished HSS tool-life data used in this study, and to Professor William G. Hunter of the University of Wisconsin for his initial suggestions about the application of power transformations to tool-life equations. The use of computing facilities in the University of Wisconsin Computing Center was supported by a grant from the University Research Committee, the University of Wisconsin.
REFERENCES


APPENDIX A

Transformations and the 95% Confidence Region

This appendix is a brief explanation of the theory of transformations for dependent and independent variables, and the determination of the 95% confidence region \([4, 5]\).

Transformations

For transformations of the observation or dependent variable \(y\) defined by

\[
y(\lambda) = \begin{cases} 
\frac{y^\lambda - 1}{\lambda} & \lambda \neq 0 \\
\ln y & \lambda = 0 
\end{cases} \tag{A-1}
\]

it is assumed that for each value of the parameter \(\lambda\), \(y(\lambda)\) is a monotonic function of \(y\). Further, it is assumed that these \(y(\lambda)\)'s are independently, normally distributed with constant variance \(\sigma^2\) and expectations

\[
\mathbb{E}[y(\lambda)] = A \theta \tag{A-2}
\]

where \(y(\lambda)\) is the column vector of transformed observations, \(A\) is the matrix of independent variables, and \(\theta\) is a column vector of unknown parameters.

To find the best value \(\hat{\lambda}\) of \(\lambda\), i.e., to find the best transformation on the \(y\)'s, the likelihood function in terms of the original \(y\)'s is considered. Since the likelihood function of the \(y(\lambda)\)'s is their joint normal density function, the likelihood in terms of the original \(y\)'s is

\[
L(\lambda; y) = (2\pi)^{-n/2} \cdot (\sigma)^{-n} \cdot \exp \left\{ \frac{(y(\lambda) - A \theta)'(y(\lambda) - A \theta)}{2 \sigma^2} \right\} \cdot J(\lambda; y) \tag{A-3}
\]

where \(n\) is the number of observations, and \(J(\lambda; y)\) is the Jacobian of the transformation and is defined by:
\[
J(\lambda;\gamma) = \frac{n}{\bar{y}} \sum_{i=1}^{n} \left| \frac{dy_i^{(\lambda)}}{dy_i} \right| = \sum_{i=1}^{n} y_i^{\lambda - 1} = (\bar{y})^{\lambda - 1} = (\gamma)^{\lambda - 1} \tag{A-4}
\]

From (A-3) the maximum likelihood estimates \( \hat{\theta} \) and \( \hat{\sigma}^2 \) can be found for a fixed \( \lambda \). These estimates are the standard least square estimates, and hence
\[
\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (y_i^{(\lambda)} - \hat{\gamma}^{(\lambda)})^2}{n} = \frac{\text{RSS}}{n} = \frac{S(\lambda;\gamma)}{n} \tag{A-5}
\]
where \( \hat{y}_i^{(\lambda)} \) is the predicted value of \( y_i^{(\lambda)} \). By substituting \( \hat{\theta} = \theta \) and \( \hat{\sigma}^2 = \sigma^2 \) into the log of the likelihood function given by (A-3), the maximized log likelihood for a fixed \( \lambda \) is
\[
\ln L_{\max}(\lambda;\gamma) = -\frac{n}{2} \ln \frac{S(\lambda;\gamma)}{n} + \ln J(\lambda;\gamma) + \text{constant} \tag{A-6}
\]

This function can be simplified, if instead of (A-1), the normalized transformation \( z(\lambda) \) is used, where
\[
z(\lambda) = \frac{\sum_{i=1}^{n} z_i^{(\lambda)}}{\sum_{i=1}^{n} z_i^{(\lambda)}} = \begin{cases} \frac{\lambda - 1}{\gamma^{\lambda - 1}} & \lambda \neq 0 \\ \gamma \ln \gamma & \lambda = 0 \end{cases} \tag{A-7}
\]
The Jacobian \( J(z;\gamma) \) of this transformation is then equal to one, and hence (A-6) reduces to
\[
\ln L_{\max}(\lambda;\gamma) = -\frac{n}{2} \ln \frac{S(\lambda;\gamma)}{n} + \text{constant} \tag{A-8}
\]
where \( S(\lambda;\gamma) \) is the RSS for the normalized dependent variable \( z^{(\lambda)} \). Thus, the value \( \lambda \) that minimizes \( S(\lambda;\gamma) \) will also maximize (A-8) and therefore \( \hat{\lambda} \) is the maximum likelihood estimate or best value of \( \lambda \).

Transformations of the independent variables \( x_i (i = 1, 2, \ldots, k) \) can be applied without affecting the assumptions of normality and constant variance, and one of the simplest transformations is the power transformation:
\[
U_i = \begin{cases} x_i^\alpha_i & \alpha_i \neq 0 \\ \ln x_i & \alpha_i = 0 \end{cases} \tag{A-9}
\]
When transformations of both the dependent and independent variables are considered together, the maximized log likelihood for a given set of transforming parameters \((\lambda, \alpha_1, \alpha_2, \ldots, \alpha_k)\) is obtained from

\[
\ln L_{\text{max}}(\lambda, \alpha_1, \alpha_2, \ldots, \alpha_k; z) = -\frac{n}{2} \ln S(\lambda, \alpha_1, \ldots, \alpha_k; z) / n + \text{constant}
\]

(A-10)

and the best values \((\hat{\lambda}, \hat{\alpha}_1, \ldots, \hat{\alpha}_k)\) of the transforming parameters are obtained by minimizing \(S(\lambda, \alpha_1, \alpha_2, \ldots, \alpha_k; z)\) with respect to \((\lambda, \alpha_1, \ldots, \alpha_k; z)\).

The 95\% Confidence Region

An approximate 100(1 - \(\gamma\))\% confidence region can be determined from

\[
\ln L_{\text{max}}(\hat{\lambda}, \hat{\alpha}_1; z) - \ln L_{\text{max}}(\lambda, \alpha_2; z) < 1/2 \chi^2_\gamma (\gamma) 
\]

(A-11)

where \(\gamma\) is the level of significance, \(\chi^2_\gamma\) is the chi-square distribution with \(\gamma\) degrees of freedom, and where \(\gamma\) is equal to the number of transforming parameters under study.

For instance, when fitting an equation of the form

\[
E[z(\lambda)] = \beta_0 + \beta_1 x_1^{\alpha_1} + \beta_2 x_2^{\alpha_2}
\]

(A-12)

for a given \(\alpha_2\) as in the body of the paper, the 100(1 - \(\gamma\))% confidence region for \(\lambda\) and \(\alpha_1\) is the area enclosed by the contour \(C_R\) where

\[
C_R = \ln L_{\text{max}}(\hat{\lambda}, \hat{\alpha}_1; z, \alpha_2) - 1/2 \chi^2_2 (\gamma)
\]

(A-13)

The corresponding contour \(C'_R\) on the RSS surface is calculated from

\[
\ln C'_R = \ln S(\hat{\lambda}, \hat{\alpha}_1; z, \alpha_2) + \chi^2_2 (\gamma) / n
\]

(A-14)

where \(S(\hat{\lambda}, \hat{\alpha}_1; z, \alpha_2)\) is the minimum RSS at the point \((\hat{\lambda}, \hat{\alpha}_1)\).
For example, the approximate 95% confidence region for $\lambda$ and $\alpha_1$ as shown in Figure 2 is determined from (A-14) as follows:

$$\ln C_R = \ln (110) + \frac{5.99}{50}$$

i.e.,

$$C_R' = e^{4.82} = 124;$$  \hspace{1cm} (A-15)

since the contour diagram is based upon an $n$ of 50 observations, the minimum RSS is 110, and at the 95th percentile the $\chi^2$ variable with two degrees of freedom is 5.99.
APPENDIX B

Sample Calculation of a RSS Value

The basis of the RSS Contour Diagram shown in the body of the paper is a grid of RSS values, where each value is determined by fitting Equation (7), i.e.,

\[ E \left[ T^{(\lambda)} \right] = \beta_0 + \beta_1 V^{\alpha_1} + \beta_2 f^{\alpha_2} \]  \hspace{1cm} (B-1)

For instance, if \( \alpha_2 = 1, \alpha_1 = 1 \) and \( \lambda = -0.2 \), then

\[ E \left[ T^{(-0.2)} \right] = E \left[ \frac{T^{-0.2}}{(-0.2)(\hat{T})^{-1.2}} \right] = \beta_0 + \beta_1 V + \beta_2 f \]  \hspace{1cm} (B-2)

The coefficients \( \beta_0, \beta_1 \) and \( \beta_2 \) are estimated by the method of least squares, i.e.,

\[ \hat{b} = (X'X)^{-1}X' \hat{y} \]  \hspace{1cm} (B-3)

where \( \hat{b} \) is the vector of estimated values of \( \hat{\beta} \)

\( X \) is the matrix of transformed independent variables

\( X' \) is the transpose of \( X \)

\( (X'X)^{-1} \) is the inverse of \( (X'X) \)

and \( \hat{y} \) is the vector of transformed observations of tool-life, i.e., \( y = T^{(\lambda)} \).

Using the carbide tool-life data [3] shown in Table B-1, the calculation of the \( \hat{b} \)'s is illustrated as follows:

1. The matrix of independent variables \( X \) consists of the values given by columns (B) (C) and (D) in Table B-1.

2. The observation vector \( \hat{y} \) consists of the values given in Column (F), and are calculated from

\[ \hat{y}_i = T_i^{(-0.2)} = \frac{T_i^{-0.2}}{(-0.2)(\hat{T})^{-1.2}} \hspace{1cm} i = 1, \ldots, n. \]

where \( T_i \) is the observed tool-life shown in column (b), and where...
\[ T = \left( T_1 \times T_2 \times T_3 \times \ldots \times T_n \right)^{\frac{1}{n}}, \text{ i.e.,} \]

\[ T = \left[ (1.75) \ (1.85) \ (2.0) \ \ldots \ (6.0) \right]^{\frac{1}{36}} = 3.692 \]

(3) By Eq. (B-3)

\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 
\end{bmatrix} =
\begin{bmatrix}
  36.466 \\
  -0.032 \\
  698.289 
\end{bmatrix}
\]

and the predicting equation is

\[ \hat{\gamma} = T^{(-0.2)} = 36.466 - 0.032 v - 698.289 f \]  \hspace{1cm} (B-4)

where \( \hat{\gamma} \) is the predicted transformed tool life.

The values of \( \gamma \) at the various cutting conditions are calculated by Eq. (B-4) and are shown in Column (G).

Finally, the residuals \( R_i \),

\[ R_i = (y_i - \hat{\gamma}_i) \quad i = 1, \ldots, n \]

are given in Column (H), the \( R_i^2 \) are given in Column (I), and thus, the value of the residual sum of squares is:

\[ \text{RSS} = \sum_{i=1}^{n} R_i^2 = 21.093. \]

This particular RSS is shown in Figure 6(\( \alpha_2 = 1 \)) at the point \( (\lambda = 0.2, \alpha_1 = 1.0) \).

By using a high speed computer, the complete RSS grid in the \( \lambda - \alpha_1 \) plane can be determined and hence RSS contours can be drawn.
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**TABLE B1**

\[ \text{RSS} = \sum R^2 = 21.033 \]
FIGURE 1

(a) $\ln V - \ln T$

(b) $V - \ln T$

(c) $V^{1/2} - \ln T$

Cutting Speed V, sfpm

Tool Life T, min.

$\n = .0014$

$\circ = .0030$

$\square = .0042$

$\times = .0228$
CAPTIONS FOR THE ILLUSTRATIONS AND TABLE

Figure 1. Cutting Speed Versus High-Speed Steel Tool Life On Three Different Sets of Scales. [6]

Figure 2. RSS Contour Diagram In The $\lambda-\alpha_1$ Plane For $\alpha_2=0$. (Tool Material - HSS, Work Material - SAE 1045, Observations - 50).

Figure 3. RSS Contour Diagram In The $\lambda-\alpha_1$ Plane For $\alpha_2=1$. (Tool Material - HSS, Work Material - SAE 1045, Observation - 50).

Figure 4. Cutting Speed Versus Carbide Tool Life On Three Different Sets of Scales [3].

Figure 5. RSS Contour Diagram In The $\lambda-\alpha_1$ Plane For $\alpha_2=0$. (Tool Material - Carbide, Work Material - SAE 4340: BHN 200, Observations - 36).

Figure 6. RSS Contour Diagram In The $\lambda-\alpha_1$ Plane For $\alpha_2=1$. (Tool Material - Carbide, Work Material - SAE 4340: BHN 200, Observations - 36).

Figure 7. RSS Contour Diagram In The $\lambda-\alpha_1$ Plane For $\alpha_2=0$. (Tool Material - Carbide, Work Material - SAE 4340: BHN 300, Observations - 30).

Figure 8. RSS Contour Diagrams In The $\lambda -\alpha_1$ Plane For $\alpha_2=1$. (Tool Material - Carbide, Work Material - SAE 4340: BHN 300, Observations - 30).

Figure 9. RSS Contour Diagrams In The $\lambda -\alpha_1$ Plane For $\alpha_2=0$. (Tool Material - Carbide, Work Material - SAE 1018, Observations - 24).

Figure 10. RSS Contour Diagram In The $\lambda -\alpha_1$ Plane For $\alpha_2=1$. (Tool Material - Carbide, Work Material - SAE 1018, Observations - 24).

Figure 11. Fitting Carbide Tool-Life Data By Four Different Transformations Over A Wide Cutting-Speed Range. [1]

Table B1 Sample Calculation of a Residual Sum of Squares (RSS) Value.