DEPARTMENT OF STATISTICS

The University of Wisconsin
Madison, Wisconsin

TECHNICAL REPORT No. 96

November, 1966

ALIAS RELATIONSHIP FOR TWO-LEVEL
PLACKETT AND BURMAN DESIGNS

by

Norman R. Draper and David M. Stoneman*

This research was partially supported by the United States Navy through the Office of Naval Research, under Contract Nonr-1202(17), Project NR042-222, and by the Wisconsin Alumni Research Foundation through the University Research Committee. Reproduction in whole or in part is permitted for any purpose of the United States Government.

*Wisconsin State University, Whitewater, Wisconsin
Alias Relationships for Two-Level
Plackett and Burman Designs

by

Norman R. Draper
University of Wisconsin, Madison, Wisconsin

David M. Stoneman
Wisconsin State University, Whitewater, Wisconsin

0. SUMMARY

Plackett and Burman (1946) provided two-level experimental designs
which allow N-1 variables to be examined in N runs where N is a multiple
of 4 between 4 and 100 (except for 92). When N is also a power of two,
the Plackett and Burman designs are identical with one or other of the
families of \(2^{k-p}\) designs (Box and Hunter, 1961a ). The alias relation-
ships for \(2^{k-p}\) type designs are easily obtained through the methods
provided by Box and Hunter. Alias relationships for the other Plackett
and Burman type designs are, however, much more complicated and not
readily available. Below we obtain the alias relationships for the N = 12
run design and provide a short University of Wisconsin Computing Center
CDC Fortran 60 program which will enable the reader to obtain other
alias relationships if he so wishes.

1. INTRODUCTION

Plackett and Burman (1946, page 323) give tables from which can be
constructed two-level designs for examining N-1 variables in N runs where
N is a multiple of four between 4 and 100 (except 92). There are three
basic construction methods:

1. Take a (specified) row of N-1 plus and minus signs. Construct
   N-2 further rows by cyclicly permuting the signs in the first
row. Add a row of all minus signs. This gives N rows (= runs) of levels for N-1 variables (=columns). The N = 12 case, used as an example below, is developed in this manner. So are the designs for N = 8, 16, 20, 24, 32, 36, 44, 48, 60, 68, 72, 80, and 84.

2. Several square blocks of plus and minus signs are given. Further rows are obtained by cyclic permutation of the blocks. A row of minus signs is added. Designs for N = 28, 52, 76, and 100 are of this type.

3. A block of plus and minus signs which we denote by A is given. The design is obtained by writing down

\[
\begin{array}{ccc}
A & A & 1 \\
A & -A & -1
\end{array}
\]

where 1 denotes a column of plus signs. This is known as "doubling" the design A, which contains only half the number of runs present in the final design. Designs for N = 40, 56, 64, 88, and 96 are obtained in this manner from the 20, 28, 32, 44, and 48 run designs, respectively.

Example: When N = 12, the design is generated from the row

\[+ + - + + - - - + -\]

and takes the following form:
Table 1: N = 12 Run Design

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Column of pluses for mean</th>
<th>Factor 1 2 3 4 5 6 7 8 9 10 11</th>
<th>Observation y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+ + - + + + - - - - +</td>
<td>$y_1$</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>- + + - + + + - - - - +</td>
<td>$y_2$</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+ - + + + + - + + - -</td>
<td>$y_3$</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>- - + - + + + - - + + + -</td>
<td>$y_4$</td>
</tr>
<tr>
<td>5</td>
<td>+</td>
<td>- - - + + - + - + + + + -</td>
<td>$y_5$</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>- - - + - + + - + + + + + -</td>
<td>$y_6$</td>
</tr>
<tr>
<td>7</td>
<td>+</td>
<td>- - - - - - + - - + + + +</td>
<td>$y_7$</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>+ + + - - - + - - + + - +</td>
<td>$y_8$</td>
</tr>
<tr>
<td>9</td>
<td>+</td>
<td>- - - - - - - - - - - - - - -</td>
<td>$y_9$</td>
</tr>
<tr>
<td>10</td>
<td>+</td>
<td>- - - - - - - - - - - - - - -</td>
<td>$y_{10}$</td>
</tr>
<tr>
<td>11</td>
<td>+</td>
<td>- - - - - - - - - - - - - - -</td>
<td>$y_{11}$</td>
</tr>
<tr>
<td>12</td>
<td>+</td>
<td>- - - - - - - - - - - - - - -</td>
<td>$y_{12}$</td>
</tr>
</tbody>
</table>

If all two-factor and higher order interactions are zero, the main effects of each of the eleven factors is estimated as follows. For factor 1, attach to the entries in the $y$ column the signs in the 1 column and divide the total by the divisor 6, which is the number of plus signs in column 1 (and in every factor column). For example the main effect of factor 1 is

$$ (y_1 - y_2 + y_3 - y_4 - y_5 - y_6 + y_7 + y_8 + y_9 - y_{10} + y_{11} - y_{12})/6 \quad (1.1) $$

The overall mean effect is obtained in similar fashion using the column of pluses and the divisor 12.

2. BIASES

We shall suppose that all two-factor interactions are not zero but that all interactions of order three or more are zero. Then the quantity
(1, 1) will not estimate the main effect of factor 1 alone but will estimate the main effect of factor 1 plus certain two-factor interactions. To see which two-factor interactions are involved with each main effect, we need to construct alias relationships. When a design is of the $2^{k-p}_{III}$ type the alias relationships are easily obtained through the methods provided by Box and Hunter (1961a). They can also be obtained through regression methods (Box, 1952). This latter approach will be used here to develop the required alias relationships for Plackett and Burman type designs which are not of the $2^{k-p}_{III}$ type.

**Obtaining biases**

Suppose we wish to fit the regression model $E(y) = X\beta$ by least squares. Provided the model is correct, the estimator $\hat{\beta} = (X'X)^{-1}X'y$ is unbiased, that is, $E(\hat{\beta}) = \beta$. If the model is not correct the estimator is biased; the extent of the bias depends not only on the postulated and true models but also on the values of the $x$-variables which enter the regression calculations. If the correct model takes the form

$$E(y) = X\beta + X_1\beta_1$$

and thus includes terms $X_1\beta_1$ which we did not consider in our estimation procedure, then it can be shown (Box, 1952), that

$$E(\hat{\beta}) = \beta + A\beta_1$$  \hspace{1cm} (2.1)

where

$$A = (X'X)^{-1}X'X_1$$

is called the alias matrix. The individual equations in (2.1) are called the alias relationships.
Application, $N = 12$

Here the fitted model will be $E(y) = X\hat{\beta}$ where

$$\hat{\beta}' = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11})$$

and where $X$ is the matrix obtained by attaching unities to the signs in the twelve columns of Table 1. These coefficients will represent half of the factorial main effects and so, for example, $\beta_1$ will be estimated by

$$(y_1 - y_2 + y_3 - y_4 - y_5 - y_6 + y_7 + y_8 + y_9 - y_{10} + y_{11} - y_{12})/12$$

where the divisor is now 12 (rather than 6 as it was in (1.1)), provided all interactions are zero. If all two-factor interactions are not zero (but all effects other than the linear and two-factor interactions effects are zero), the model $E(y) = X\hat{\beta}$ is not correct and the correct model is of the form $E(y) = X\hat{\beta} + X_1\hat{\beta}_1$ where

$$\hat{\beta}_1' = (\beta_{12}, \beta_{13}, \beta_{14}, \ldots, \beta_{10}, 11)$$

and where $X_1$ is a 12 x 55 matrix and is obtained as follows: Obtain a "12" column by placing the product of signs in a row of columns 1 and 2, Table 1, in the same row of the "12" column. Obtain a "13" column by placing the product of the signs in a row of columns 1 and 3, Table 1, in the same row of the "13" column. Do this for all 55 of the two-factor interactions, and then attach unities to the signs obtained. The following are the first three columns of the $X_1$ matrix for this example:
Two-factor interactions

<table>
<thead>
<tr>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>+1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>+1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>+1</td>
<td>+1</td>
<td>+1</td>
</tr>
</tbody>
</table>

For general $N$ the matrix $(X'X)^{-1}$ will have value $\frac{1}{N} I$ where $I$ is a unit matrix of size $N \times N$. The matrix $X'_i X_i$, which is the transpose of the matrix $X'_i X_i$ which we require, is easily obtained using the program in the appendix. When $N = 12$ we obtain the following:
The alias structure in terms of regression coefficients is now obtained by reading down the columns of $X'_i X$, dividing all entries by $N = 12$ and attaching them to the corresponding interaction coefficient indicated in the extreme left hand column. For example from the column headed "1" we see that (2.2) provides us with an estimate not of $\beta_1$ but of

$$\beta_1 + (1/3)(-\beta_{23} - \beta_{24} - \beta_{25} + \beta_{26} - \cdots - \beta_{10,11}).$$

All other alias relationships are obtained in similar fashion. We see that these relationships are very complicated; the only coefficients not aliased with $\beta_1$ are $\beta_{ij}$ where $j \neq 1$. In general, then, the Plackett and Burman designs are quite unsuitable for situations when two-factor interactions are expected to occur and are most useful as "main-effect designs". If two-factor interactions are expected, then the designs should be "folded over" whereupon main effects become clear of all two-factor interactions. This principle is described by Box and Hunter (1961a).

REFERENCES


APPENDIX: THE PROGRAM

The program given below took 35 seconds on the CDC 1604 to obtain the alias relationships given above. The program can be used for other values of $N \leq 100$. The input shown consists of $12 (=N)$ followed by the $X'$ matrix with the first row of unities deleted.
Plackett and Burman (1946) provided two-level experimental designs which allow \(N-1\) variables to be examined in \(N\) runs where \(N\) is a multiple of 4 between 4 and 100 (except for 92). When \(N\) is also a power of two, the Plackett and Burman designs are identical with one or other of the families of \(2^{k-p}\) designs (Box and Hunter, 1961a). The alias relationships for \(2^{III}\) type designs are easily obtained through the methods provided by Box and Hunter. Alias relationships for the other Plackett and Burman type designs are, however, much more complicated and not readily available. Below we obtain the alias relationships for the \(N = 12\) run design and provide a short University of Wisconsin Computing Center CDC Fortran 60 program which will enable the reader to obtain other alias relationships if he so wishes.