Technical Report No. 92

October, 1966

INVESTIGATION OF REJECTION RULES FOR OUTLIERS IN SMALL SAMPLES FROM THE NORMAL DISTRIBUTION. III. ESTIMATION OF THE VARIANCE (KNOWN MEAN).

by

Irwin Guttman and Dennis Smith*

* Smith currently at HRB Singer, Inc., State College, Pa.

This work was partially supported by the National Science Foundation and the Wisconsin Alumni Research Foundation, October 1966.
Investigation of Rejection Rules for Outliers in Small Samples from the Normal Distribution. III: Estimation of the Variance (Known Mean).

by

Irwin Guttman and Dennis Smith

0. Summary

In this report we investigate the performance of rejection rules when concerned with estimating the variance $\sigma^2$ of a normal distribution $N(\mu, \sigma^2)$ with known mean $\mu$. We shall assume that we have a sample $(y_1, \ldots, y_n)$ of independent observations, hopefully each $y_i$ from $N(\mu, \sigma^2)$, but

where perhaps one of the observations may be spurious, arising from either $N(\mu + a\sigma, \sigma^2)$ or $N(\mu, (1+b)\sigma^2)$. 
1. **Introduction**

When concerned with estimating $\sigma^2$, the A, W, and S-rules have the general form defined in (5.2.2), (5.3.2), and (5.4.2) of Report #90. In this report, we obtain the necessary expressions for the premiums and protections in sections 2.1, 2.2, and 2.3. Exact formulas for odd $n$ are obtained for the A and S-rules, and results of the calculations are given in sections 3.5 and 4.4. For the W-rule when $n>3$, calculations using Monte Carlo were necessary. A description is given in section 4.1.
2. Functional Form of Premiums and Protections

For a given rejection region and a rejection rule which uses
the estimator \( \hat{\sigma}^2 \), say, which is unbiased in the null case, we may rewrite
the expressions for premium and protection given in section 4 of Report #90
as

\[
\text{Premium} = \frac{V(\hat{\sigma}^2) - V(s^2)}{V(s^2)} \quad (2.1)
\]

when all observations are from \( N(\mu, \sigma^2) \), and where \( s^2 \) is given by

\[
s^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu)^2 \quad (2.2)
\]

Also, we have seen that

\[
\text{Protection} = \frac{E(s^2 - \sigma^2) - E(\hat{\sigma}^2 - \sigma^2)^2}{E(s^2 - \sigma^2)^2} \quad (2.3)
\]

defining a spurious observation is present.

Since \( \mu \) is known, we shall assume that \( \mu = 0 \), and thus we
deal with the \( N(0, \sigma^2) \), \( N(\alpha \sigma, \sigma^2) \), and \( N(0, (1+b) \sigma^2) \) distributions.

2.1 Premiums

When all \( n \) observation are from \( N(0, \sigma^2) \), \( s^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2 \)
is distributed as \( \frac{\sigma^2}{n} \chi_n^2 \), where \( \chi_n^2 \) denotes a chi-square variate with
\( n \) degrees of freedom. Hence,

\[
V(s^2) = \frac{\sigma^4}{n^2} V(\chi_n^2) = \frac{2\sigma^4}{n} \]
leaving only \( V(\hat{\sigma}^2) \) to be computed in order to obtain the premium for a given rule.

Thus, for a rejection rule with estimator \( \hat{\sigma}^2 \), the premium is, from (2.1)

\[
\text{Premium} = \frac{nV(\hat{\sigma}^2)}{2\sigma^4} - 1.
\]

Since we are dealing with rejection rules which use only unbiased estimators, we have

\[
\text{Premium} = \frac{nE(\hat{\sigma}^2)}{2\sigma^4} - \frac{n+2}{2} \quad (2.4)
\]

Since \( \mu = 0 \), if we define \( \hat{y}_i^2 = y_i, (i=1,\ldots,n) \), and order these as \( \hat{y}(1)^2 < \cdots < \hat{y}(n) \), we see from Report #90 that the estimators corresponding to the rejection rules under consideration are of the form

\[
\hat{\sigma}^2 = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} \hat{y}(i) & \text{if } \hat{y}(n) < L \sum_{i=1}^{n-1} \hat{y}(i) \\
Dg(\hat{y}(1)^2, \ldots, \hat{y}(n)) & \text{if } \hat{y}(n) \geq L \sum_{i=1}^{n-1} \hat{y}(i) 
\end{cases} \quad (2.5)
\]

with \( L = \frac{K}{n-K} \), where \( K \) is the rejection constant given in Report #90, and where, for the A-Rule, W-Rule, and the S-Rule, respectively, we find after consulting (5.2.2), (5.3.2), and (5.4.2) of Report #90, that

\[
g(h) = g(h(1), \ldots, h(n)) = \frac{1}{n-1} \sum_{i=1}^{n-1} h(i) \quad (2.6)
\]

\[
g(h) = g(h(1), \ldots, h(n)) = \frac{1}{n-1} \left[ \sum_{i=1}^{n-1} h(i) + h(n-1) \right] \quad (2.7)
\]

\[
g(h) = g(h(1), \ldots, h(n)) = \frac{1}{n} \left[ \sum_{i=1}^{n-1} h(i) + \frac{L}{1+L} \sum_{i=1}^{n} h(i) \right] \quad (2.8)
\]
Suppose that, for a given rejection rule and a given \( L \), we wish to compute the corresponding premium. First of all, we must determine the value of \( D \) needed in the rejection rule. This is accomplished by solving the following equation for \( D \):

\[
\sigma^2 = E(\hat{\sigma}^2) = \text{Prob} \left( h(n) < L \sum_{i=1}^{n-1} h(i) \right) E \left[ \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right] E \left[ \frac{D^2}{n} \sum_{i=1}^{n-1} h(i) \right] \\
+ \text{Prob} \left( h(n) \geq L \sum_{i=1}^{n-1} h(i) \right) E \left[ Dg(h) - \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right] E \left[ h(n) \geq L \sum_{i=1}^{n-1} h(i) \right]
\]

Since \( E \left[ \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right] = DE(\hat{\sigma}^2) = D\sigma^2 \), the above equation is seen to be equivalent to

\[
\sigma^2 = D\sigma^2 \\
+ \text{Prob} \left( h(n) \geq L \sum_{i=1}^{n-1} h(i) \right) E \left[ Dg(h) - \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right] E \left[ h(n) \geq L \sum_{i=1}^{n-1} h(i) \right]
\]

(2.9)

From (2.4) we see that, having found \( D \), we must compute \( E(\hat{\sigma}^N) \) in order to evaluate the premium. We may simplify this computation somewhat by noting that

\[
E(\hat{\sigma}^N) = \text{Prob} \left( h(n) < L \sum_{i=1}^{n-1} h(i) \right) E \left[ \left\{ \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right\}^2 \right] E \left[ h(n) < L \sum_{i=1}^{n-1} h(i) \right] \\
+ \text{Prob} \left( h(n) \geq L \sum_{i=1}^{n-1} h(i) \right) E \left[ \left( Dg(h) \right)^2 \right] E \left[ h(n) \geq L \sum_{i=1}^{n-1} h(i) \right]
\]
and since \( \mathbb{E} \left[ \frac{D}{n} \sum_{i=1}^{n} h_i \right]^{2} = D^2 \mathbb{E}(s^4) = \frac{(n+2) D^2 \sigma^4}{n} \),

we have that

\[
\mathbb{E}(\hat{\sigma}^4) = \frac{(n+2) D^2 \sigma^4}{n} \\
+ \text{Prob } (h_{(n)} \geq L \sum_{i=1}^{n-1} h_i) \mathbb{E} \left[ \left( \frac{Dg(h)}{n} \right)^2 - \left( \frac{D}{n} \sum_{i=1}^{n} h_i \right)^2 \right] \\
\left| h_{(n)}^2 \sum_{i=1}^{n-1} h_i \right|
\]

(2.10)

which we substitute into (2.4) in order to obtain the premium.

It should be noted that iteration may be used to determine

the constants \( L \) and \( D \) corresponding to a given premium. We see, however,

that we must solve two equations, one arising from the requirement of

unbiasedness, and the other from the given value of the premium.

2.2. Protections---Biased Mean Case

Since \( \mu = 0 \), we may consider \( y_i \sim N(0, \sigma^2), i = 1, \ldots, n-1 \), and

\( y_n \sim N(a\sigma, \sigma^2), -\infty < a < \infty \), where the \( y_i \)'s are independent. Since the

rejection rules to be examined make no use of the assumption that \( y_n \) is

the spurious observation, this assumption does not cause loss of generality.
Hence, we have that \( s^2 \sim \chi_n^2 \frac{n}{n} (\chi_{n-1}^2 + \chi_1^2 \alpha^2) \), where \( \chi_1^2, \alpha^2 \) denotes a non-central chi-square variate with one degree of freedom and non-centrality parameter \( \alpha^2 \).

Thus,

\[
E(s^2) = \frac{(n+\alpha^2) \sigma^2}{n} 
\tag{2.11}
\]

and

\[
V(s^2) = \frac{2(n+2\alpha^2) \sigma^4}{n^2} 
\tag{2.12}
\]

We have, then,

\[
E(s^4) = V(s^2) + E^2(s^2) = \frac{\sigma^4}{n^2} \left[ 2(n+2\alpha^2) + (n+\alpha^2)^2 \right]
\]

which gives us

\[
E(s^2 - \sigma^2)^2 = E(s^4) - 2\sigma^2 E(s^2) + \sigma^4 
= \frac{\sigma^4}{n^2} (2n + 4\alpha^2 + \alpha^4) 
\tag{2.13}
\]

Thus, to determine the protection afforded by a given rule using estimator \( \hat{\sigma}^2 \), say, we need to calculate \( E(\hat{\sigma}^2 - \sigma^2)^2 \), and together with (2.13) substitute into (2.3).

Following the steps outlined in section 2.1, we find that we may write

\[
E(\hat{\sigma}^2 - \sigma^2)^2 = E\left( \frac{D}{n} \sum_{i=1}^{n} h(i) - \sigma^2 \right)^2 
+ \text{Prob} \left( h(n) \geq \sum_{i=1}^{n-1} h(i) \right) E\left( \left[ \frac{D}{n} \sum_{i=1}^{n} h(i) - \sigma^2 \right]^2 - \left[ \frac{D}{n} \sum_{i=1}^{n} h(i) - \sigma^2 \right]^2 \right)
\tag{2.14}
\]
where, of course, $y_n$, and hence $h_n$, is the spurious observation. Using (2.11) and (2.13), we have

\[
E \left[ \frac{D}{n} \sum_{1}^{2} h(i) - \sigma^2 \right]^2 = E \left[ D s^2 - \sigma^2 \right]^2
\]

\[
= D^2 (s^2 - \sigma^2)^2 + 2\sigma^2 D(D-1) E(s^2 - \sigma^2) + (D-1)^2 \sigma^4
\]

\[
= \frac{\sigma^4}{n^2} \left[ D^2(2n + 4a^2 + a^4) + 2n D(D-1) a^2 + n^2(D-1)^2 \right]
\]

Substituting this result into (2.14) and (2.3), we have that when a spurious observation with mean shifted by $a\sigma$ is present in the sample, the protection afforded by a given rejection rule using estimator $\hat{\sigma}^2$ is

\[
\text{Protection} = \frac{N_1(a) + N_2(a)}{E(s^2 - \sigma^2)^2}
\]

(2.15)

where

\[
N_1(a) = \frac{\sigma^4}{n^2} \left[ (1-D^2) (2n+4a^2+a^4) - 2nD(D-1) a^2 + n^2(D-1)^2 \right]
\]

(2.16)

\[
N_2(a) = \text{Prob}(h(n) \geq L \sum_{1}^{n-1} h(i)) E \left\{ \left[ \frac{D}{n} \sum_{1}^{2} h(i) - \sigma^2 \right]^2 \right\}
\]

\[
\left. \left[ \left[ h(n) \geq L \sum_{1}^{n-1} h(i) \right] \right. \right]
\]

(2.17)

and $E(s^2 - \sigma^2)^2$ is given by (2.13).
2.3. Protests—Biased Variance Case

Since \( \mu = 0 \), consider \( y_i \sim N(0, \sigma^2) \), \( i = 1, \ldots, n-1 \), and \( y_n \sim N(0, (1+b) \sigma^2) \), \( 0 \leq b \), where the \( y_i \)'s are independent. Again, the assumptions cause no loss of generality.

We have, in this case, that \( s^2 \sim \frac{\sigma^2}{n} (\chi^2_{n-1} + (1+b) \chi^2_1) \).

Thus,

\[
E(s^2) = \frac{(n+b) \sigma^2}{n} \tag{2.18}
\]

\[
V(s^2) = \frac{2(n+2b+b^2)}{n^2} \sigma^4 \tag{2.19}
\]

\[
E(s^4) = V(s^2) + E^2(s^2) = \frac{\sigma^4}{n^2} \left[ \frac{2(n+2b+b^2)}{n^2} + (n+b)^2 \right]
\]

and

\[
E(s^2 - \sigma^2)^2 = \frac{\sigma^4}{n^2} (2n+4b+3b^2) \tag{2.20}
\]

Therefore, for a given rejection rule using estimator \( \hat{\sigma}^2 \), we need to calculate \( E(\hat{\sigma}^2 - \sigma^2)^2 \) and substitute this and (2.20) into (2.3) in order to obtain the protection provided by the rule when a spurious observation with an inflated variance, specifically \( (1+b) \sigma^2 \), is present in the sample.

In a manner similar to that of the previous section, we find that for a given bias "\( b \)" in the variance,

\[
\text{Protection} = \frac{M_1(b) + M_2(b)}{E(s^2 - \sigma^2)^2} \tag{2.21}
\]
where

\[ M_1(b) = \frac{\sigma^4}{n^2} \left[ (1-D^2)(2n+4b+3b^2) - 2nD(D-1)b - n^2(D-1) \right] \]  \hspace{1cm} (2.22)

\[ M_2(b) = \text{Prob}(h_{(n)} \geq L \sum_{l=1}^{n} h_{(l)}) \times \mathbb{E} \left[ \left( \frac{D}{n} \sum_{l=1}^{n} h_{(l)} - \sigma^2 \right)^2 - [\text{Dg}(h) - \sigma^2]^2 \right] \right|_{h_{(n)} \geq L \sum_{l=1}^{n} h_{(l)}} \]  \hspace{1cm} (2.23)

and \( \mathbb{E}(\sigma^2 - \sigma^2)^2 \) is given by (2.20).

3. **Analysis and Distribution Theory \((n=3)\)**

We now consider the special case of a sample size of three. We discuss the determination of required constants for a given rule and premium in section 3.1, and the computation involved in obtaining protections in sections 3.2 and 3.3.
3.1. Determining the Rejection Constants for a Given Premium (n=3)

In the null case, we have three independent observations
\[ y_1, y_2, y_3, \] each from \( N(0, \sigma^2) \), where \( \sigma^2 \) is to be estimated. Thus, we shall be considering the random variables \( h_i = y_i^2, (i = 1, 2, 3) \) and the ordered \( h_i \)'s, \( h_1 < h_2 < h_3 \).

The joint density of \( (h_1, h_2, h_3) \) is given by

\[
p(h_1, h_2, h_3) = \frac{6}{(2\pi)^{3/2} \sigma^3} (h_1, h_2, h_3)^{1/2} \exp \left[ -\frac{1}{2\sigma^2} (h_1 + h_2 + h_3) \right]
\]

\[ (3.1) \]

\[ 0 < h_1 < h_2 < h_3 \]

We shall use this density quite extensively in the calculation of premiums.

Let us define the region

\[
R = \{ (h_1, h_2, h_3) \mid h_3 \geq L (h_1 + h_2) \}
\]

\[ (3.2) \]

To compute, for the A-Rule, the premium corresponding to a given \( L \), we must first solve equation (2.9) for \( D \). Thus, for \( n = 3 \), we must solve the following for \( D \):

\[
\sigma^2 = D \sigma^2 + D \int_R \left[ \frac{h_1 + h_2}{6} - \frac{h_3}{3} \right] p(h_1, h_2, h_3) \Pi dh(i)
\]

\[ (3.3) \]
and

\[ E(\hat{\sigma}_S^4) = \frac{5}{3} D^2 \sigma^4 \]

\[ + \frac{D^2}{9(1+L)^2} \int_R \left[ (3L^2 + 2L)(h_{(1)} + h_{(2)})^2 + 2(L^2 - L - 1)h_{(3)}(h_{(1)} + h_{(2)}) \right. 

\[ - (2L + 1) h_{(3)}^2 \left. \right] \prod \{h_{(1)}h_{(2)}h_{(3)}\} \prod h_{(i)} \]

As mentioned above, the constants which result in a given premium for a specific rule may be found by iteration. Section 3.5 lists each rule considered and the corresponding rejection constants for premiums of 5%, 4%, 3%, 2%, 1% and .5%.

3.2. Calculation of Protection (n=3) --- Biased Mean Case

In this section we derive the computational form of the protections given by the rejection rules when a spurious observation from a normal distribution with biased mean is present in a sample of three observations. Numerical results are given in section 3.5.

If we are using a particular rejection rule and have obtained the required constants for a given premium, we may use the formulas of section 2.2 to calculate protection when an observation, which we may take to be \( y_3 \) without loss of generality, is from \( N(\mu + a\sigma, \sigma^2) \).
Thus, with \( \mu = 0 \), we have the independent variables

\[
y_1 \sim N(0, \sigma^2)
\]
\[
y_2 \sim N(0, \sigma^2)
\]
\[
y_3 \sim N(a\sigma, \sigma^2)
\]

As in section 3.1, we shall define \( h_i = y_i^2 \), \( i=1, 2, 3 \) and order the \( h_i \)'s as \( h_1 < h_2 < h_3 \).

Thus, the density of \( h_1 \) (or \( h_2 \)) is

\[
f(h_1; \sigma) = (2\pi \sigma^2)^{-\frac{1}{2}} \exp \left( -\frac{h_1}{2\sigma^2} \right) \quad (3.5)
\]

and the density of \( h_3 \) is

\[
g(h_3; a, \sigma) = \exp \left( -\frac{a^2}{2} \right) \cosh \left( \frac{h_3 a}{\sigma} \right) f(h_3; \sigma) \quad (3.6)
\]

Of course, \( g(h_3; a, \sigma) \) is the density of a \( a \chi^2_1, a^2 \) variate.

Hence, the density of \( (h_1, h_2, h_3) \) is, for a given bias "a",

\[
p(h_1, h_2, h_3; a) = 2f(h_1; \sigma)f(h_2; \sigma)g(h_3; a, \sigma)
\]
\[
+ 2f(h_1; \sigma)g(h_2; a, \sigma)f(h_3; \sigma)
\]
\[
+ 2g(h_1; a, \sigma)f(h_2; a, \sigma)f(h_3; \sigma)
\]
\[
(3.7)
\]

over the region

\[
0 < h_1 < h_2 < h_3
\]

Thus, using this density, we may calculate the protection by evaluating the function \( N_2(a) \) defined by (2.17), and substituting into (2.15).
3.3 Calculation of Protection (n=3)---Biased Variance Case

We indicate, in this section, the computational form of protections when a spurious observation from a normal distribution with inflated variance is present in a sample of three observations. Numerical results are given in section 3.5.

Without loss of generality, we may assume that we are sampling on \((y_1, y_2, y_3)\), where the \(y_i\)'s are independent, and

\[
y_1 \sim N(0, \sigma^2)
\]

\[
y_2 \sim N(0, \sigma^2)
\]

\[
y_3 \sim N(0, (1+b) \sigma^2)
\]

As in the previous section, we shall consider the variables \(h_i\) and \(h_{(i)}\) \((i=1, 2, 3)\). In the present case, the density of \(h_3\) is

\[
r(h_3; b, \sigma) = \left[2\pi h_3 (1+b) \sigma^2 \right]^{-\frac{1}{2}} \exp \left[ \frac{-h_3^2}{2(1+b) \sigma^2} \right]
\]

(3.8)

and the density of \(h_i\), \((i=1, 2)\), is \(f(h_i; \sigma)\), as given by (3.5).

Thus, the density of \((h_{(1)}, h_{(2)}, h_{(3)})\), for a given bias "b" in the variance, is

\[
q(h_{(1)}, h_{(2)}, h_{(3)}; b) = 2f(h_{(1)}; \sigma)f(h_{(2)}; \sigma)r(h_{(3)}; b, \sigma) + 2f(h_{(1)}; \sigma)r(h_{(2)}; b, \sigma)f(h_{(3)}; \sigma) + 2r(h_{(1)}; b, \sigma)f(h_{(2)}; \sigma)f(h_{(3)}; \sigma)
\]

(3.9)

over

\[0 < h_{(1)} < h_{(2)} < h_{(3)}.\]
In order to compute the protection afforded by a given rule, we must first evaluate the function \( M_2(b) \), given by (2.23). We do this, of course, by using the density of \( q(h_1, h_2, h_3; b) \), given by (3.9).

3.4. Method of Computation (n=3)

Since the calculations needed to evaluate protections are fairly involved, we shall indicate the form of these calculations in this section. As will be noted, calculations of this type are not necessary to evaluate protections for the A- Rule or the S- Rule, since the more simple method of section 4 may be used. Hence, we shall concentrate on the evaluation of the protection afforded by the W- Rule in the biased mean case. The calculations for the other rules are similar, as are those for the biased variance case.

To compute the protection given by the W- Rule when an observation from \( N(u + \sigma, \sigma^2) \) is present in the sample, we need to evaluate \( N_2(a) \), given by (2.17). Using (2.7) and (3.7), we may write \( N_2(a) \), for \( n=3 \), as

\[
N_2(a) = \int_R \left[ \frac{D^2}{9} (h_3^2 - 3h_{(2)}^2 + 2h_{(1)}h_3 + 2h_{(2)}h_3 - 2h_{(1)}h_{(2)}) + \frac{2\sigma^2 D}{3} (h_{(2)} - h_{(3)}) \right] p(h_{(1)}, h_{(2)}, h_{(3)}; a) \prod h_{(1)}
\]

where \( R \) is the region defined by (3.2).
Let us now consider the transformation

\[ w = h_1 \]

\[ v = h_{(1)} + h_{(2)} \]

\[ q = \frac{h_{(3)}}{h_{(1)} + h_{(2)}} \]  \hspace{1cm} (3.10)

A rationale for this transformation is that, in the null case, \( q \) will be a "pseudo-\( F \)" variate.

If, for the premiums of interest, the rejection constant \( L \) is such that \( L \geq 1 \) (assumed a priori, confirmed a posteriori), then \( N_2(a) \) may be written as

\[ N_2(a) = \frac{b^2}{g} \left( 4I_1 + I_3 + 3I_4 - I_5 \right) + \frac{2a^2}{3} D(I_1 - I_2 - I_3) \]  \hspace{1cm} (3.11)

with

\[ I_1 = \int_T vf(w, v, q; a) \, dw \, dv \, dq \]

\[ I_2 = \int_T w f(w, v, q; a) \, dw \, dv \, dq \]

\[ I_3 = \int_T v q f(w, v, q; a) \, dw \, dv \, dq \]

\[ I_4 = \int_T v^2 f(w, v, q; a) \, dw \, dv \, dq \]

\[ I_5 = \int_T v^2 q f(w, v, q; a) \, dw \, dv \, dq \]

\[ I_6 = \int_T v^2 q^2 f(w, v, q; a) \, dw \, dv \, dq \]
\[ I_7 = \int_T vW(w,v,q; a) \, dwvdq \]

\[ I_8 = \int_T w^2 f(w,v,q; a) \, dwvdq \]

where \( T \) is the region

\[ T = \{ (w,v,q) \mid 0 < w < \frac{v}{2}, \ 0 < v < \infty, \ L < q < \infty \}, \]

and where

\[ f(w,v,q; a) = \]

\[
\frac{2}{(2\pi)^{3/2} \sigma^3} v^{1/2} w^{-1/2} (v-w)^{-1/2} q^{-1/2} \exp \left[ -\frac{1}{2\sigma^2} (v+qv+a^2\sigma^2) \right] \]

\[
\sum_{j=0}^{\infty} \frac{(a^2 \sigma^2 q)^j}{(2j)!} \int_0^\infty \frac{a^2 (v-w)^j}{o^{2j}} + \frac{a^2 w^j}{0 (2j)! o^{2j}} \right] \}
\]

(3.12)

over the region

\[ \{ (w,v,q) \mid \max \left[ 0, v(1-q) \right] < w < \frac{v}{2}, \ 0 < v < \infty, \ \frac{1}{2} < q < \infty \}. \]

If we define the functions

\[ C(r) = \frac{1}{2v^r} \int_0^{v/2} w^{\frac{1}{2}} (v-w)^{-\frac{1}{2}} \frac{1}{2} \, dw \]

\[ S(r) = \frac{1}{2v^r} \int_0^{v/2} w^{r-\frac{1}{2}} (v-w)^{-\frac{1}{2}} \, dw \]

and

\[ R(L; r, m) = \int_0^\infty q^{-\frac{1}{2}} (l+q)^{-r-m-\frac{1}{2}} \, dq, \]
we may evaluate the integrals $I_1, \ldots, I_8$ by using the recurrence relations

$$C(r+1) = \frac{1}{2(r+1)} \left[ (2r+1) C(r) + \left( \frac{1}{2} \right)^{r+1} \right]$$

$$C(0) = \frac{\pi}{4}$$

$$S(r+1) = \frac{1}{2(r+1)} \left[ (2r+1) S(r) - \left( \frac{1}{2} \right)^{r+1} \right]$$

$$S(0) = \frac{\pi}{4}$$

and

$$R(L; r, m+1) = \frac{2}{(2r+2m+1)} \left[ mR(L; r, m) - m - \frac{1}{2} \right]$$

$$R(L; r, 1) = \frac{2}{2r+1} \left[ 1 - L \left( \frac{1}{2} \right)^{r+\frac{1}{2}} \right]$$

Needless, to say, the algebra involved in evaluating the required integrals is horrendous.

3.5. Tables and Graphs of Results (n=3)

The following tables and graphs present the numerical results obtained for the case n=3. All calculations were carried out on the CDC 1504 computer.

Rejection constants corresponding to premiums of 5%, 4%, 3%, 2%, 1%, and .5% and protections corresponding to premiums of 5% and 1% are tabulated. The graphs present protections corresponding to premiums of 5% and 1%. 
<table>
<thead>
<tr>
<th>S-Rule</th>
<th>W-Rule</th>
<th>A-Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.98908</td>
<td>74.8967</td>
<td>0.0200</td>
</tr>
<tr>
<td>3.69402</td>
<td>37.95963</td>
<td>0.00401</td>
</tr>
<tr>
<td>2.66267</td>
<td>16.42028</td>
<td>0.00804</td>
</tr>
<tr>
<td>2.08628</td>
<td>12.39766</td>
<td>1.0209</td>
</tr>
<tr>
<td>1.68376</td>
<td>92.62331</td>
<td>1.0166</td>
</tr>
<tr>
<td>1.61357</td>
<td>73.90780</td>
<td>1.0025</td>
</tr>
</tbody>
</table>

Rejection constants corresponding to a given premium for a sample of size three.

**Table 3.5.1**
Figures 3.5.1 and 3.5.2

Projections corresponding to premiums of 5% and 10% when a spurious observation from $N(\mu, \sigma^2)$ is present in a sample of size three. (Symmetric about $\bar{X} = 0$.)
<table>
<thead>
<tr>
<th>A-Rule</th>
<th>S-Rule</th>
<th>A-Rule</th>
<th>S-Rule</th>
<th>A-Rule</th>
<th>S-Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.39</td>
<td>0.123</td>
<td>0.129</td>
<td>0.490</td>
<td>0.483</td>
<td>0.499</td>
</tr>
<tr>
<td>0.38</td>
<td>0.111</td>
<td>0.119</td>
<td>0.485</td>
<td>0.472</td>
<td>0.493</td>
</tr>
<tr>
<td>0.225</td>
<td>0.080</td>
<td>0.086</td>
<td>0.478</td>
<td>0.341</td>
<td>0.363</td>
</tr>
<tr>
<td>0.283</td>
<td>0.061</td>
<td>0.065</td>
<td>0.468</td>
<td>0.271</td>
<td>0.295</td>
</tr>
<tr>
<td>0.066</td>
<td>0.041</td>
<td>0.054</td>
<td>0.453</td>
<td>0.240</td>
<td>0.260</td>
</tr>
<tr>
<td>0.390</td>
<td>0.030</td>
<td>0.036</td>
<td>0.425</td>
<td>0.213</td>
<td>0.230</td>
</tr>
<tr>
<td>0.206</td>
<td>0.017</td>
<td>0.024</td>
<td>0.384</td>
<td>0.141</td>
<td>0.159</td>
</tr>
<tr>
<td>0.189</td>
<td>0.010</td>
<td>0.013</td>
<td>0.340</td>
<td>0.084</td>
<td>0.096</td>
</tr>
<tr>
<td>0.101</td>
<td>0.002</td>
<td>0.003</td>
<td>0.208</td>
<td>0.030</td>
<td>0.043</td>
</tr>
<tr>
<td>0.031</td>
<td>0.006</td>
<td>0.009</td>
<td>0.050</td>
<td>0.030</td>
<td>0.040</td>
</tr>
</tbody>
</table>

**Proportions for a 1% premium**

**Proportions for a 5% premium**

From \( N(y + \delta, \sigma) \) to present in a sample of size three.

Proportions corresponding to premium of 5% and 1% when a spurious observation

Table 3.5.2
<table>
<thead>
<tr>
<th>Table 3.5.3</th>
<th>Protection for a 5% Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A-Rule</td>
</tr>
<tr>
<td>1.02</td>
<td>0.45</td>
</tr>
<tr>
<td>1.04</td>
<td>0.46</td>
</tr>
<tr>
<td>1.06</td>
<td>0.47</td>
</tr>
<tr>
<td>1.08</td>
<td>0.48</td>
</tr>
<tr>
<td>1.10</td>
<td>0.49</td>
</tr>
</tbody>
</table>

From $\theta^* = (1+\theta)^2$ the present is a sample of size three.
4. **Analysis (n>3)**

In the following sections we discuss the calculations of rejection constants and protections corresponding to a desired premium for sample sizes greater than three. As will be noted, the computations for the A-Rule and S-Rule may be performed analytically if the rejection constant $L_{21}$. Thus, to avoid Monte Carlo, we shall consider premiums which are such that this condition holds.

When dealing with the W-Rule, it appears that any analytic procedure fails, due to the form of the rule and the unwieldy regions over which integration must be performed. Thus, we indicate Monte Carlo procedures which may be used to determine, or rather, estimate, desired rejection constants and protections. Unfortunately, these procedures are not as efficient as those given when considering estimation of the mean (Report # 91), and require a large amount of computer time to obtain "decent" bounds on the Monte Carlo approximations. We shall therefore content ourselves, for the W-Rule, with determining only the rejection constants corresponding to the premiums considered. Hopefully, a Monte Carlo that is less time consuming than the one suggested or, perhaps, even tractable analytic computations, will be found to make evaluation of the W-Rule an easier task.

The framework of our problem consists, of course, of a sample of $n$ from a $N(0, \sigma^2)$ distribution, hopefully, but where one of the observations may be from a $N(a\sigma, \sigma^2)$ or a $N(0, (1+b) \sigma^2)$ distribution.

Again let the sample of $n$ be denoted by $(y_1, \ldots, y_n)$, let $h_i = y_i^2$, $(i=1, \ldots, n)$. Denote the ordered $h_i$'s by $h_{(1)}<\ldots<h_{(n)}$, and define

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2 = \frac{1}{n} \sum_{i=1}^{n} h_i = \frac{1}{n} \sum_{i=1}^{n} h_{(i)}$$
4.1. Determining Rejection Constants for a Given Premium \( n>3 \)

Since the calculations when considering odd sample sizes turn out to be simpler than when considering even sample sizes, we shall restrict ourselves to \( n=5, 7, 9, 11 \). Also, we shall see that the techniques presented for the A-Rule and the S-Rule may be used to advantage for \( n=3 \).

Assuming that \( L \) is given, we find from (2.9) and (2.6) that for the A-Rule we need to solve the following equation for \( D \):

\[
s^2 = D s^2 + \text{Prob}\left( h_{\text{zL}} \sum_{l=1}^{n-1} h_l \right) E\left[ \frac{D}{n-1} \sum_{l=1}^{n-1} h_l - \frac{D}{n} \sum_{l=1}^{n} h_l \right] h_{\text{zL}} \sum_{l=1}^{n-1} h_l
\]

If \( L>1 \), we see that

\[
h_{n} \geq L \sum_{l=1}^{n-1} h_l \Rightarrow h_{\text{zL}} = h_n
\]

and thus, using symmetry, the above equation is equivalent to

\[
s^2 = D s^2 + n \text{Prob}\left( h_{n} \geq L \sum_{l=1}^{n-1} h_l \right) E\left[ \frac{D}{n-1} \sum_{l=1}^{n-1} h_l - \frac{D}{n} \sum_{l=1}^{n} h_l \right] h_{n} \geq L \sum_{l=1}^{n-1} h_l
\]

which we may rewrite as

\[
s^2 = D s^2 + \sigma_n^2 \text{Prob}\left( x_{1} \geq L x_{2} \right) E\left[ \frac{D x_2}{n-1} - \frac{D(x_1+x_2)}{n} \right] x_{1} \geq L x_{2}
\]

(4.1)

where \( x_1 \sim x_1^2 \) and \( x_2 \sim x_{n-1}^2 \).
Thus, we see we need only evaluate the expectation in the right hand side of (4.1), that is,
\[
\frac{n}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} 2^{n/2} \int_0^\infty \int_0^L x_1^{1/2} \left[ \frac{x_2}{n(n-1)} - \frac{x_1}{n} \right] x_1^{n-3} \exp\left[\frac{1}{n}x_1 + x_2\right] dx_2 dx_1
\]

Since we are restricting ourselves to odd values of \( n \), we may evaluate the inner integral by means of the recurrence relation
\[
\int x^N e^{-\frac{x}{2}} dx = -2xe^{-\frac{x}{2}} + 2N \int x^{N-1} e^{-\frac{x}{2}} dx
\]
and the outer integral collapses to a weighted sum of gamma functions.

From (2.10) and (2.6), we find for the A-Rule and estimator \( \hat{\sigma}^2_A \)
\[
E(\hat{\sigma}^2_A) = \frac{n+2}{n} D^2 \sigma^4 + \text{Prob} (h(n) \geq L \sum_{i=1}^{n-1} h(i)) E \left[ (\frac{D}{n-1})^2 \sum_{i=1}^{n-1} h(i)^2 \left| \sum_{i=1}^{n-1} h(i) \geq L \sum_{i=1}^{n-1} h(i) \right. \right] - (\frac{D}{n})^2 \left. (\sum_{i=1}^{n-1} h(i)^2) \right| h(n) \geq L \sum_{i=1}^{n-1} h(i)
\]

which, if \( L \geq 1 \), we may rewrite as
\[
E(\hat{\sigma}^2_A) = \frac{n+2}{n} \sigma^4 + n\sigma^4 \text{Prob} (x_1 \geq L x_2) E \left[ \frac{D^2 x_2^2}{(n-1)^2} - \frac{D^2 (x_1 + x_2)^2}{n^2} \left| x_1 \geq L x_2 \right. \right]
\]

Thus, we may also evaluate \( E(\hat{\sigma}^2_A) \) analytically by means of the recurrence relation given above, and may then substitute the value of \( E(\hat{\sigma}^4_A) \) into (2.4) to obtain the premium corresponding to the given \( L \), when using the A-Rule.

As noted earlier, this procedure cannot be used to investigate the W-Rule. For, even with \( L \geq 1 \), we must keep track of the value of \( h(n-1) \).
We turn, then, to Monte Carlo in an attempt to solve the equations of interest, which from \((2.7), (2.9),\) and \((2.10)\) are

\[
\sigma^2 = D \sigma^2 \quad \text{and} \quad E(\sigma^2) = \frac{n+2}{n} D \sigma^2
\]

\[
+ \text{Prob}(h(n) \geq L \sum_{i=1}^{n-1} h(i)) \mathbb{E} \left[ \frac{D}{n} (h(n) - h(n-1)) \mid h(n) \geq L \sum_{i=1}^{n-1} h(i) \right]
\]

and

\[
E(\sigma^4) = \frac{n+2}{n} D \sigma^4
\]

\[
+ \text{Prob}(h(n) \geq L \sum_{i=1}^{n-1} h(i)) \mathbb{E} \left[ \frac{D}{n} (h(n) + h(n-1)) \mid h(n) \geq L \sum_{i=1}^{n-1} h(i) \right]^2 - \left( \frac{D}{n} \sum_{i=1}^{n-1} h(i) \right)^2
\]

\[
\left. \mid h(n) \geq L \sum_{i=1}^{n-1} h(i) \right\}
\]

One way to approach these equations is to assume \(\sigma^2 = 1,\) i.e., to consider "estimating" \(1,\) and generate \(h(1), \ldots, h(n)\) subject to the condition that \(h(n) \geq L \sum_{i=1}^{n-1} h(i).\) Assuming \(L \geq 1,\) we see that

\[
\text{Prob}(h(n) \geq L \sum_{i=1}^{n-1} h(i)) = n \text{Prob}(h_n \geq L \sum_{i=1}^{n-1} h_i) = n \text{Prob}(z_1 \geq n-1 L)
\]

\[(4.3)\]

where \(z_1\) is an \(F_1, n-1\) variate. Let us denote the density of \(z_1\) by \(f(z_1)\) and the density of \(z_1\) given that \(z_1 \geq (n-1) L\) by

\[
f(z_1 \mid z_1 \geq (n-1) L) = \left[ \int_{(n-1) L}^{\infty} f(z_1) dz_1 \right]^{-1} f(z_1) \quad (n-1) L < z_1 < \infty.
\]
If we now obtain a random number \( u \in (0, 1) \), we may solve

\[
u = \int_{(n-1)L}^{Q} f(z_1 | z_1 \geq (n-1)L) \, dz_1
\]

for \( q \), the corresponding percentage point. For odd \( n \), we may use the helpful fact that

\[
\int_{a}^{b} f(z_1) \, dz_1 = \frac{2}{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{n})} \int_{w_1}^{w_2} \cos^{n-2} x \, dx
\]

where \( w_1 = \tan^{-1}(\frac{a}{n-1})^\frac{1}{2} \) and \( w_2 = \tan^{-1}(\frac{b}{n-1})^\frac{1}{2} \).

Having determined \( q \), we are interested in choosing \( h_1, \ldots, h_n \)

subject to the restriction

\[
h_n = \frac{q}{n-1} \sum_{i=1}^{n-1} h_i
\]

If we consider

\[
z_1 = \frac{(n-1)h_n}{\sum_{i=1}^{n-1} h_i}
\]

and

\[
z_2 = \sum_{i=1}^{n-1} h_i
\]

we find that the conditional density of \( z_2 \), given that \( z_1 = q \), is that of a

\[
\chi^2_{n} \quad \text{variate.}
\]

Hence, we take an observation, say \( d \), on this variate, whereby we obtain

\[
\sum_{i=1}^{n-1} h_i = d, \quad \text{and} \quad h_n = \frac{qd}{n-1}
\]

The individual \( h_i \)'s and \( h_{(i)} \)'s, \( (i=1, \ldots, n-1) \), may be obtained by choosing a random angle in \( (n-1) \)-space and finding its intersection with the sphere of radius \( d_{\frac{1}{2}} \).
This procedure may be repeated for a given L for N sets of \( h_i \)'s, with the desired rejection constants found by iteration.

As noted above, we may determine

\[
p_1 = \text{Prob} \left( h(n) \geq L \sum_{1}^{n-1} h(i) \right)
\]

analytically. Let us denote, for the j-th set of \( h_i \)'s generated by our procedure, the functions

\[
k_j = \frac{h(n-1)_j - h(n)_j}{n}
\]

and

\[
t_j = \left( \frac{1}{n} \left( \sum_{1}^{n-1} h(i) + h(n-1)_j \right) \right)^2 - \left( \frac{1}{n} \sum_{1}^{n} h(i) \right)^2
\]

Further, let us define

\[
\theta_1 = E \left[ \frac{1}{n} (h(n-1)_j - h(n)_j) \left| h(n) \geq L \sum_{1}^{n-1} h(i) \right. \right]
\]

\[
\theta_2 = E \left[ \left( \frac{1}{n} \left( \sum_{1}^{n-1} h(i) + h(n-1)_j \right) \right)^2 - \left( \frac{1}{n} \sum_{1}^{n} h(i) \right)^2 \left| h(n) \geq L \sum_{1}^{n-1} h(i) \right. \right]
\]

\[
\hat{\theta}_1 = \frac{1}{N} \sum_{1}^{N} k_j
\]

and

\[
\hat{\theta}_2 = \frac{1}{N} \sum_{1}^{N} t_j
\]

Thus, we have that the Monte Carlo estimate of \( D \) is

\[
\hat{D} = \frac{1}{1 + p_1 \hat{\theta}_1}
\]

and, using (2.4), the estimate of the premium is

\[
\hat{p} = \frac{n}{2(1 + p_1 \hat{\theta}_1)^2} \left( \frac{n+2}{n} + p_1 \hat{\theta}_2 \right) - \frac{n+2}{2}
\]
Using a Taylor series expansion truncated at first order terms, we have that the estimated variances of $\hat{D}$ and $\hat{p}$ are, approximately,

$$\hat{V}(\hat{D}) = \frac{p_1^2 \hat{V}(\hat{\theta}_1)}{(1+p_1 \hat{\theta}_1)^4}$$

and

$$\hat{V}(\hat{p}) = \left[ \frac{n p_1}{(1+p_1 \hat{\theta}_1)^3} \left( \frac{n+2}{n} + p_1 \hat{\theta}_2 \right) \right]^2 \hat{V}(\hat{\theta}_1)$$

$$+ \frac{n^2 p_1^2}{4(1+p_1 \hat{\theta}_1)^3} \hat{V}(\hat{\theta}_2)$$

$$- \frac{n^2 p_1^2}{(1+p_1 \hat{\theta}_1)^5} \frac{n+2}{n} + p_1 \hat{\theta}_2 \right) \text{Cov}(\hat{\theta}_1, \hat{\theta}_2)$$

Using these quantities, we may construct (approximate) confidence intervals for the premium and the rejection constant $D$.

The value of $N$ used in these calculations is based on the size of the estimated standard error of the estimated premium, $\hat{p}$. Specifically, the rejection constants $L$ and $D$ are determined in the same manner that the constant $C$ was found when concerned with estimating the mean (see section 4.1 of Report # 91).

Turning to the S-Rule and estimator $\hat{\sigma}_s^2$ we see that if $L > 1$, we may apply the same procedure used in investigating the A-Rule. Thus, from (2.8), (2.9), and (2.10) we find we must solve

$$\sigma^2 = D \sigma^2$$

$$+ \sigma_n^2 \text{Prob}(x_1 \geq L x_2) \left[ \frac{D}{n(l+L)} \left( L x_2 - x_1 \right) \left| x_1 \geq L x_2 \right. \right]$$
\[ E(\hat{\sigma}^4) = \frac{n+2}{n} \sigma^4 + n\sigma^4 \text{Prob} (x_1^2 \geq x_2) E \left[ \frac{D}{n^2} \left( \frac{(2L+1)^2}{(L+1)^2} - 1 \right) \frac{x_2}{x_1^2} + \frac{2L(2L+1)}{(L+1)^2} - 1 \right] x_1 x_2 + \left[ \frac{L}{(L+1)^2} \right] x_1^2 \mid x_1^2 \geq x_2 \] 

where, as before,

\[ x_1^2 \sim \chi_1^2 \]

and

\[ x_2^2 \sim \chi_{n-1}^2 \]

Hence, we may use recurrence relation (4.2) to evaluate the integrals which arise from these equations.

4.2. Calculation of Protections (n>3)—Biased Mean Case

In this section we consider the sample of n independent observations \((y_1, \ldots, y_n)\) where \(y_i \sim N(0, \sigma^2), \ (i = 1, \ldots, n-1)\), and where \(y_n \sim N(a\sigma, \sigma^2)\). To compute protection for a given rejection rule and given rejection constants, we must evaluate (2.17). When considering the A-Rule, we may, by using (2.6) and the symmetry of \(h_1, \ldots, h_{n-1}\), rewrite (2.17) as

\[ N_2(a) = T_1(a) + T_2(a) \]
with

\[ T_1(a) = \sigma^4 \text{Prob}(w_1 \geq Lw_2) E \left\{ \left[ \frac{D(w_1 + w_2)}{n} - 1 \right]^2 - \left[ \frac{Dw_2}{n-1} - 1 \right]^2 \right| w_1 \geq Lw_2 \right\} \]

(4.4)

where

\[ w_1 \sim x_1^2, \ a^2, \ w_2 \sim x_{n-1}^2 \]

and

\[ T_2(a) = \sigma^4(n-1) \text{Prob}(t_1 \geq Lt_2) E \left\{ \left[ \frac{D(t_1 + t_2)}{n} - 1 \right]^2 - \left[ \frac{Dt_2}{n-1} - 1 \right]^2 \right| t_1 \geq Lt_2 \right\} \]

(4.5)

where

\[ t_1 \sim x_1^2, \ t_2 \sim x_{n-1}^2, \ a^2 \]. The term \( T_1(a) \) may be written in integral form as

\[ T_1(a) = \frac{\sigma^4}{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2})} 2^{n/2} \int_0^\infty \int_0^\infty \left\{ \left[ \frac{D(w_1 + w_2)}{n} - 1 \right]^2 - \left[ \frac{Dw_2}{n-1} - 1 \right]^2 \right\} \]

\[ \cdot \left( w_1^{-1/2} \ w_2^{-n/2} \exp \left[ - \frac{1}{2} (a^2 + w_1 + w_2) \right] \sum_{j=0}^{\infty} \frac{(a^2 w_1)^j}{(2j)!} dw_2 dw_1 \right) \]

For odd values of \( n \geq 3 \), assuming that the order of integration and summation may be interchanged, we find that evaluating \( T_1(a) \) involves integrals of the form

\[ \int_0^\infty \int_0^L w_1 w_2 \left( \frac{m-1}{2} \right) \exp \left[ - \frac{1}{2} (w_1 + w_2) \right] dw_2 dw_1 \]
where $k$ and $m$ are integers. Applying recurrence relation (4.2) to the inner integral, we are left with integrals of the type

$$
\int_{0}^{\infty} \frac{r-1}{w_1^2} \exp(-dw_1) \, dw_1
$$

where $r$ is an integer, and $d$ is a positive quantity. Hence, $T_1(a)$ may be reduced to an infinite sum of weighted gamma functions.

The term $T_2(a)$ may be expressed as

$$
T_2(a) = \frac{\sigma^4(n-1)}{2^n/\pi} \int_{0}^{1} \int_{0}^{t_1} \left\{ \left[ \frac{B(t_1+t_2)}{n} - 1 \right]^2 - \left[ \frac{D_2}{n-1} - 1 \right]^2 \right\} t_1^{\frac{1}{2}} t_2^{\frac{n-3}{2}} \exp \left[ -\frac{1}{2} (a_1^2 + t_1 + t_2) \right] \sum_{j=0}^{\infty} \frac{(a_1^2 t_2)^j}{(2j)!} \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\frac{n-1}{2})} \, dt_2 \, dt_1
$$

which may also be reduced to an infinite sum of weighted gamma functions by use of the procedure indicated above.

Since the S-Rule may be handled in a similar fashion, the work is omitted here.

As noted previously, the above procedure cannot be used to investigate the W-Rule, and it appears that a good Monte Carlo technique is called for. Unfortunately, all the procedures we have attempted present the problem of requiring an exhorbitant amount of computer time. One of these procedures is to parallel that used in the null case. However, this means inverting the distribution function of a non-central $F$ variate, and choosing points at random on a hypersphere not centered at the origin, which are no small tasks. We have not considered protections for the W-Rule because
of these drawbacks, but hope that in the future more efficient means will be found to evaluate the effect of this rule.

4.3. **Calculation of Protections (n>3)—Biased Variance Case**

We now turn to the situation where the \( n \) independent observations \( (y_1, \ldots, y_n) \) are such that \( y_i \sim N(0, \sigma^2) \), \( (i = 1, \ldots, n-1) \), and where \( y_n \sim N(0, (1+b) \sigma^2) \).

Suppose we consider the A-Rule with given rejection constants, and wish to compute the protection afforded by the rule. To do this we must evaluate \( M_2(b) \), defined by (2.23). Using (2.6) and the symmetry of \( h_1, \ldots, h_{n-1} \), we may write \( M_2(b) \) as

\[
M_2(b) = W_1(b) + W_2(b)
\]

with

\[
W_1(b) = \sigma^4 \text{Prob} \left[ (1+b) x_1 \geq L x_2 \right] E \left[ D \left( \frac{(1+b)x_1 + x_2}{n} \right)^2 - 1 \right] - \left( \frac{D x_2}{n-1} - 1 \right)^2 \left( (1+b) x_1 \geq L x_2 \right)
\]

where

\[
x_1 \sim \chi_1^2, \quad x_2 \sim \chi_{n-1}^2
\]
\[ W_2(b) = \sigma^4(n-1) \text{Prob} \left( z_1 \geq Lz_2 + L(1+b)z_3 \right) \]
\[ E \left[ \frac{D[z_1 + z_2 + (1+b)z_3]}{n} \cdot \left( 1 - \frac{1}{n-1} \right)^2 \right] \]
\[ \left| z_1 \geq Lz_2 + L(1+b)z_3 \right| \]

where
\[ z_1 \sim x_1^2, \quad z_2 \sim x_{n-2}^2, \quad z_3 \sim x_1^2 \]

Now, \( W_1(b) \) may be written as
\[ W_1(b) = \frac{\sigma^4}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} 2^{n/2} \int_0^L \int_0^{x_1} \int_0^{x_2} \exp \left[ -\frac{1}{2} (x_1 + x_2) \right] dx_2 dx_1 \]

which may be evaluated by using the methods of the preceding section.

If we consider the transformation
\[ z_1 = t_1 t_3 \]
\[ z_2 = t_2 t_3 \]
\[ z_3 = t_3 \]

we may write \( W_2(b) \) in integral form as
\[ W_2(b) = \frac{(n-1) \sigma^4}{n \Gamma\left(\frac{n-2}{2}\right) 2^{n/2}} \int_0^L \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \exp \left[ -\frac{1}{2} t_3 (t_1 + t_2 + 1+b) \right] \]
\[ dt_3 dt_2 dt_1 \]
Thus, integrating out $t_3$, we are left with having to evaluate integrals of the form
\[
\int_{a_1}^{a_2} \int_{0}^{t_1} \int_{t_2}^{t_1} (t_1^2 + t_2^2 + 1)^{-r/2} dt_2 dt_1
\]
where $n$ is an odd integer, $p$, $q$, and $r$ are odd integers.

By considering the transformation
\[
x = \tan^{-1} \left( \frac{t_2}{(t_1 + 1)^2} \right)
\]
this double integral may be reduced to a single integral of a sum of terms involving $t_1$. This resulting integral may be evaluated by numerical integration.

The computations of protections under the S-Rule are similar, so we shall not discuss them here.

As far as the W-Rule is concerned, analytic methods such as those given above cannot be used, and our search for an efficient Monte Carlo procedure has reached an impasse, due to difficulties similar to those noted in section 4.2.

4.4. Tables and Graphs of Results ($n>3$)

The procedures outlined above were programmed for the CDC 1604 computer. The numerical results obtained are indicated in the following graphs and tables. The premiums considered vary for different values of $n$. As mentioned in section 4, this is to avoid Monte Carlo computation where possible. Specifically, the cases considered are $n=5$ and premiums of 5% and 1%, $n=7$ and premiums of 3% and 1%, $n=9$ and premiums of 1% and .5%, $n=11$ and a premium of
.5%. Also, when considering protections, we have investigated only the A-Rule and S-Rule, due to the lack of a technique for handling the W-Rule efficiently.
In order to conserve computer time, the constants for the W-rule were determined by Monte Carlo. Since we have not investigated the predictions afforded by this rule, we have centered ourselves with moderate accuracy of the reflection constants. In Table 4.4.1:

<table>
<thead>
<tr>
<th>T</th>
<th>1.000</th>
<th>1.000</th>
<th>2.000</th>
<th>3.000</th>
<th>4.000</th>
<th>5.000</th>
<th>6.000</th>
<th>7.000</th>
<th>8.000</th>
<th>9.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1.000</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
<td>9.000</td>
</tr>
<tr>
<td>T</td>
<td>1.000</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
<td>9.000</td>
</tr>
<tr>
<td>T</td>
<td>1.000</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
<td>9.000</td>
</tr>
<tr>
<td>T</td>
<td>1.000</td>
<td>1.000</td>
<td>2.000</td>
<td>3.000</td>
<td>4.000</td>
<td>5.000</td>
<td>6.000</td>
<td>7.000</td>
<td>8.000</td>
<td>9.000</td>
</tr>
</tbody>
</table>

Refection Constants

Table 4.4.1
<table>
<thead>
<tr>
<th>S-Rule</th>
<th>A-Rule</th>
<th>S-Rule</th>
<th>A-Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.49</td>
<td>6.49</td>
<td>0.654</td>
<td>9.62</td>
</tr>
<tr>
<td>0.60</td>
<td>7.54</td>
<td>0.922</td>
<td>9.77</td>
</tr>
<tr>
<td>0.47</td>
<td>9.49</td>
<td>0.858</td>
<td>8.77</td>
</tr>
<tr>
<td>0.48</td>
<td>0.32</td>
<td>0.624</td>
<td>7.62</td>
</tr>
<tr>
<td>0.99</td>
<td>2.34</td>
<td>0.602</td>
<td>6.62</td>
</tr>
<tr>
<td>0.33</td>
<td>1.31</td>
<td>0.369</td>
<td>5.64</td>
</tr>
<tr>
<td>0.77</td>
<td>3.87</td>
<td>5.00</td>
<td>4.36</td>
</tr>
<tr>
<td>0.20</td>
<td>0.41</td>
<td>0.988</td>
<td>3.16</td>
</tr>
<tr>
<td>0.09</td>
<td>0.01</td>
<td>0.259</td>
<td>2.04</td>
</tr>
<tr>
<td>0.04</td>
<td>0.06</td>
<td>0.75</td>
<td>1.03</td>
</tr>
</tbody>
</table>

Protections for a 7.5% Premium

Protections corresponding to premiums of 5% and 10% when a spurious observation from

\( N \left( \bar{x}, 0^2 \right) \) is present in a sample of size 5.

Table 4.4.2
<table>
<thead>
<tr>
<th>Rule</th>
<th>A-Rule</th>
<th>S-Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>-0.07</td>
<td>0.082</td>
</tr>
<tr>
<td>1.24</td>
<td>0.06</td>
<td>0.25</td>
</tr>
<tr>
<td>0.33</td>
<td>0.04</td>
<td>0.16</td>
</tr>
<tr>
<td>0.48</td>
<td>0.01</td>
<td>0.14</td>
</tr>
<tr>
<td>0.67</td>
<td>0.0</td>
<td>0.12</td>
</tr>
<tr>
<td>0.86</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.09</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0</td>
<td>0.08</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0</td>
<td>0.07</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0</td>
<td>0.06</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0</td>
<td>0.05</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0</td>
<td>0.04</td>
</tr>
<tr>
<td>2.2</td>
<td>0.0</td>
<td>0.03</td>
</tr>
<tr>
<td>2.4</td>
<td>0.0</td>
<td>0.02</td>
</tr>
<tr>
<td>2.6</td>
<td>0.0</td>
<td>0.01</td>
</tr>
<tr>
<td>2.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.4</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.6</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4.4</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4.6</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.4</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.6</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.4</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.6</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>7.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Protections for a 1% premium

Protections for a 3% premium

Protections corresponding to premiums of 3% and 1% when a spurious observation from M(H + 400) is present in a sample

Table 4.4.3
<table>
<thead>
<tr>
<th>A-Rule</th>
<th>S-Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 97</td>
<td>9 67</td>
</tr>
<tr>
<td>5 83</td>
<td>9 18</td>
</tr>
<tr>
<td>5 63</td>
<td>8 25</td>
</tr>
<tr>
<td>5 34</td>
<td>6 79</td>
</tr>
<tr>
<td>4 88</td>
<td>4 95</td>
</tr>
<tr>
<td>4 44</td>
<td>3 07</td>
</tr>
<tr>
<td>3 10</td>
<td>1 56</td>
</tr>
<tr>
<td>1 87</td>
<td>0 68</td>
</tr>
<tr>
<td>0 90</td>
<td>0 11</td>
</tr>
<tr>
<td>0 34</td>
<td>-0 04</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.4**

*Protection for a 5% premium of size 9.*

*From $N(N + a, \sigma^2)$ is present in a sample*.

*Protection corresponding to premiums of 1% and 5% when a spurious observation.*

*Table 4.4*
<table>
<thead>
<tr>
<th>Value</th>
<th>Value</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.661</td>
<td>0.991</td>
<td>0.110</td>
</tr>
<tr>
<td>0.946</td>
<td>0.722</td>
<td>0.9</td>
</tr>
<tr>
<td>0.626</td>
<td>0.919</td>
<td>0.8</td>
</tr>
<tr>
<td>0.549</td>
<td>0.808</td>
<td>0.7</td>
</tr>
<tr>
<td>0.644</td>
<td>0.626</td>
<td>0.6</td>
</tr>
<tr>
<td>0.404</td>
<td>0.506</td>
<td>0.5</td>
</tr>
<tr>
<td>0.344</td>
<td>0.206</td>
<td>0.4</td>
</tr>
<tr>
<td>0.204</td>
<td>0.073</td>
<td>0.3</td>
</tr>
<tr>
<td>0.101</td>
<td>0.012</td>
<td>0.2</td>
</tr>
<tr>
<td>0.039</td>
<td>-0.004</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Bias "n" = 4.5

*(n + 80, 0.05) is present in a sample of size 11.

Pretects corresponding to a premum of 0.5% when a spurious observation from

Table 4.4.5
Table 4.4.6

Protections corresponding to premiums of 5% and 1% when a spurious observation from $N(\mu, (1+b) \sigma^2)$ is present in a sample of size 5.

<table>
<thead>
<tr>
<th>Bias &quot;b&quot;</th>
<th>Protections for a 5% Premium</th>
<th>Protections for a 1% Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A-Rule</td>
<td>S-Rule</td>
</tr>
<tr>
<td>1</td>
<td>-.021</td>
<td>.053</td>
</tr>
<tr>
<td>2</td>
<td>.054</td>
<td>.207</td>
</tr>
<tr>
<td>3</td>
<td>.143</td>
<td>.321</td>
</tr>
<tr>
<td>4</td>
<td>.229</td>
<td>.398</td>
</tr>
<tr>
<td>6</td>
<td>.378</td>
<td>.489</td>
</tr>
<tr>
<td>8</td>
<td>.495</td>
<td>.539</td>
</tr>
<tr>
<td>10</td>
<td>.586</td>
<td>.569</td>
</tr>
<tr>
<td>12</td>
<td>.656</td>
<td>.589</td>
</tr>
<tr>
<td>14</td>
<td>.711</td>
<td>.603</td>
</tr>
<tr>
<td>16</td>
<td>.756</td>
<td>.614</td>
</tr>
</tbody>
</table>
5. Discussion of Results

Of the three rules we have investigated for guarding against a spurious observation when estimating $\sigma^2$ with $\mu$ known, we need consider only the A-Rule and S-Rule for a sample size of three, since the protections given by the W-Rule are dominated by those given by the A-Rule. Whether this situation also exists for larger sample sizes is, of course, an open question, but it appears that we have made no extraordinary sacrifice by not evaluating the effect of the W-Rule for sample sizes larger than three.

For a sample size of three, it may be noted that the values of $L$ listed for the A-Rule and W-Rule are extremely large compared to the corresponding values of $L$ for larger sample sizes. The reason for this is that we are, roughly speaking, dealing with an $F_{1, n-1}$ variate, and as may be seen from any table of the $F$ distribution, the percentage points rapidly decrease as $n$ increases from 3 to 5, and then gradually level off.

For the S-Rule, the relatively small values of $L$ for even a sample of size three reflect the fact that this rule retains, in the estimator, part of the suspected outlying observation.

In regard to performance, the S-Rule affords better protection than the A-Rule for small biases, with the opposite holding for large biases. Although we have attempted to assist the experimenter in his choice of a rejection rule by constructing our graphs and tables, the ultimate decision rests with him.

Similar to the suggestion in section 5 of Report # 91, a possible next step in the investigation of rejection rules for estimating $\sigma^2$ with $\mu$ known might be the consideration of a composite A-W-S-Rule, or perhaps, only a composite A-S-Rule.
Bibliography
