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DISTRIBUTION OF RESIDUAL AUTOCORRELATIONS
IN INTEGRATED AUTOREGRESSIVE-MOVING
AVERAGE TIME SERIES MODELS

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1. Introduction

An approach to the modeling of stationary and nonstationary time series such as commonly occur in economic situations and control problems is discussed by Box and Jenkins [3] and involves iterative use of the three-stage process of identification, estimation, and diagnostic checking. Given a discrete time series $z_t$, $z_{t-1}$, $z_{t-2}$, ... and using $B$ for the backward shift operator such that $Bz_t = z_{t-1}$, the general integrated autoregressive moving-average model of order $(p, d, q)$ discussed in [3] may be written

$$\phi(B) v^d z_t = \theta(B) a_t \quad (1.1)$$

with $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$ and with $\{a_t\}$ a sequence of independent normal deviates with common variance $\sigma_a^2$, to be referred to as white noise. In some instances the model will be appropriate after a suitable transformation is made on $z$.

This general class of models is too rich to allow immediate fitting to a particular sample series $\{z_t\} = z_1, z_2, \ldots, z_n$ and the following strategy is therefore employed:

1. A process of identification is used to find a smaller subclass of models worth considering to represent the stochastic process;
2. A model in this subclass is fitted by efficient statistical methods;
3. An examination of the adequacy of the fit is made.

The object of the third or diagnostic checking stage is not merely to determine whether there is evidence of lack of fit but also to suggest ways in which the model may be modified when this is necessary. Two basic methods for doing this are suggested.
(a) **Overfitting.** The model may be deliberately over-parametrized in a way it is feared may be needed and in a manner such that the entertained model is obtained by setting certain parameters in the more general model at fixed values, usually zero. One can then check the adequacy of the original model by fitting the more general model and considering whether or not the additional parameters could reasonably take on the specified values appropriate to the simpler model.

(b) **Diagnostic checks applied to the residuals.** The method of overfitting is most useful where the nature of the alternative feared model is known. Unfortunately, this information may not always be available, and less powerful but more general techniques are needed to indicate the way in which a particular model might be wrong. It is natural to consider the stochastic properties of the residuals \( \hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \) calculated from the sample series using the model (1) with estimates \( \hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_p; \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_q \) substituted for the parameters. In particular their autocorrelation function

\[
\hat{r}_k = \frac{\sum \hat{a}_t \hat{a}_{t-k}}{\sum \hat{a}_t^2}
\]

may be studied.

Now if the a's for the particular sample series were calculated using the true parameter values, then if the model were appropriate they would be uncorrelated random deviates, and their autocorrelations

\[
r_k = \frac{\sum a_t a_{t-k}}{\sum a_t^2}
\]

would for moderate or large \( n \) possess a multivariate normal distribution [1]. Also it can readily be shown that

\[
V (r_k) = \frac{n-k}{n(n+2)} \simeq \frac{1}{n},
\]

\[
\text{Cov} (r_k, r_{k+j}) = 0, \ j \neq 0.
\]
It follows that if we took a particular set of $m$ of these autocorrelations, say $r_1, r_2, \ldots, r_m$, then

$$n(n+2) \sum_{k=1}^{m} \frac{1}{n-k} r_k^2$$

would for large $n$ be distributed as $\chi^2$ with $m$ degrees of freedom; or as a further approximation,

$$n \sum_{k=1}^{m} r_k^2 \sim \chi^2_m.$$ (1.6)

It is tempting to suppose that these same properties might to a sufficient approximation be enjoyed by the $\hat{r}$'s from the fitted model; and diagnostic checks based on this supposition were suggested by Box and Jenkins [3] and Box, Jenkins, and Bacon [4]. If this assumption were warranted, approximate standard errors of $\frac{1}{n} \sqrt{n(n+2)}$ [or more accurate standard errors of $\sqrt{n-k} \frac{1}{n(n+2)}$] could be attached to the $\hat{r}$'s and a quality-control-chart type of approach used, with particular attention being paid to the $\hat{r}$'s of low order for the indication of possible model inadequacies. Also it might be supposed that eq. (1.6) with $\hat{r}$'s replacing $r$'s would still be approximately valid, so that large values of this statistic would place the model under suspicion.

It was pointed out by Durbin [6], however, that the approximation referred to above might be a very poor one. For example he showed that $\hat{r}_1$ calculated from the residuals from a first order autoregressive process could have a much smaller variance than $r_1$ for white noise.

The present paper therefore considers in some detail the properties of the $\hat{r}$'s and in particular their covariance matrix. This is done with the intention of obtaining a suitable modification to the above diagnostic checking procedures.
2. Distribution of Residual Autocorrelations

For the Autoregressive Process

The general autoregressive process of order \( p \) may be written

\[
\phi(B) y_t = a_t
\]  

(2.1)

where \( B, \phi(B) \), and \( \{a_t\} \) are as in (1.1). This model can also be expressed as a moving average process of infinite order by writing

\[
\psi(B) = \phi^{-1}(B) = (1 - \phi_1 B - \ldots - \phi_p B^p)^{-1}
\]

\[
= (1 + \psi_1 B + \psi_2 B^2 + \ldots).
\]

Then

\[
y_t = \psi(B) a_t, \quad \text{and} \quad (2.2)
\]

\[
(1 - \phi_1 B - \ldots - \phi_p B^p)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = 1 \quad (2.3)
\]

Since the coefficient of \( B^v \) is zero, \( v \geq 1 \), the \( \psi \)'s and \( \phi \)'s satisfy the relation

\[
\psi_v = \left\{ \begin{array}{ll}
\phi_1 \psi_{v-1} + \ldots + \phi_{v-1} \psi_1 + \phi_v, & v \leq p \\
\phi_1 \psi_{v-1} + \ldots + \phi_p \psi_{v-p}, & v \geq p.
\end{array} \right.
\]

Therefore by setting \( \psi_v = 0 \) for \( v < 0 \), we have

\[
\psi_0 = 1; \quad \phi(B) \psi_v = 0, \quad v \neq 0. \quad (2.4)
\]

Suppose then we have a series \( \{y_t\} \) generated by the model (2.1) or (2.2), where in general \( y_t = v^dz_t \) will be the \( d \)th difference (\( d = 0, 1, 2, \ldots \)) of the actual observations. Then for given values \( \hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_p)' \) of the parameters we can
define

\[ \dot{a}_t = a_t(\hat{\phi}) = \phi(B) y_t \]

\[ = y_t - \phi_1 y_{t-1} - \ldots - \phi_p y_{t-p} \]  \hspace{1cm} (2.5)

and the corresponding autocorrelation

\[ r_k = r_k(\hat{\phi}) = \frac{\sum \dot{a}_t \dot{a}_{t-k}}{\sum \dot{a}_t^2} \] \hspace{1cm} (2.6)

Thus, in particular,

(i) \[ a_t(\hat{\phi}) = a_t \text{ as in } (2.1), (2.2); \]

(ii) \[ a_t(\hat{\phi}) = \hat{a}_t \text{ are the residuals when } \]

(2.1) is fitted and least squares estimates \( \hat{\phi} \) obtained;

and

(iii) \[ r_k = \frac{\sum a_t a_{t-k}}{\sum a_t^2} , \quad \hat{r}_k = \frac{\sum \hat{a}_t \hat{a}_{t-k}}{\sum \hat{a}_t^2} \] \hspace{1cm} (2.8)

are the white noise and residual autocorrelations.

**Linear constraints on the \( \hat{r} \)'s.**

It is known that the residuals \( \{\hat{a}_t\} \) in (2.7) satisfy the orthogonality conditions

\[ \sum \hat{a}_t y_{t-j} = 0, \quad 1 \leq j \leq p. \] \hspace{1cm} (2.9)

Therefore if we let

\[ \hat{\psi}(B) = \hat{\phi}^{-1}(B) = (1 - \hat{\phi}_1 B - \ldots - \hat{\phi}_p B^p)^{-1} \]

then
\[ y_t = \hat{\psi}(B) \hat{a}_t = \hat{a}_t + \hat{\psi}_1 \hat{a}_{t-1} + \hat{\psi}_2 \hat{a}_{t-2} + \ldots, \quad (2.10) \]

and from (2.9) we have

\[ 0 = \sum_{t} a_t [1 + \hat{\psi}_1 B + \hat{\psi}_2 B^2 + \ldots] \hat{a}_{t-j} \]

\[ = \sum_{t} \sum_{k} \hat{\psi}_k \hat{a}_t \hat{a}_{t-k-j} \]

\[ = \sum_{k} \hat{\psi}_k \hat{r}_{k+j} \]

\[ = \sum_{k=0}^{\infty} \psi_k \hat{r}_{k+j} + \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1}) \hat{r}_{k+j} \]

\[ = \sum_{k=0}^{\infty} \psi_k \hat{r}_{k+j} + o(\frac{1}{n}). \quad (2.11) \]

In leading up to (2.11) we have presumably summed an infinite number of autocorrelations from a finite series. However since \( \{ y_t \} \) is stationary we have \( \psi_k \to 0 \) as \( k \) becomes large; and unless \( \phi \) is extremely close to the boundary of the stationarity region, this dying out of \( \psi_k \) is fast, so that the summation can generally be stopped at a value of \( k \) much less than \( n \). More precisely, we are assuming that \( n \) is large enough so that there exists a number \( m \) where

(i) all \( \psi_j \) where \( j \geq m - p \) are of order \( \frac{1}{\sqrt{n}} \) or smaller;

(ii) the ratio \( m/n \) is itself of order \( \frac{1}{\sqrt{n}} \).

Then in (2.11) and in all following discussion the error in stopping the summations at \( k = m \) can to the present degree of approximation be ignored; and (ii) also ensures that "end effects" (Such as there being only \( n - k \) terms summed in the numerator of \( \hat{r}_k \) compared with \( n \) terms in the denominator) can also be neglected.
2.1 Theoretical distribution of \( \hat{r} \).

In this section we obtain the joint distribution of \( \hat{r} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_m)' \) for an autoregressive process (2.1) of order p. This is done by expanding \( \hat{r}_k \) about \( r_k \), showing that the \( \psi \)'s in (2.2) or (2.4) closely approximate the derivatives in this expansion, and then using (2.11) to obtain a linear relationship between \( r \) and \( \hat{r} \) analogous to that between the errors and the residuals in a standard regression situation.

Expansion of \( \hat{r}_k \) about \( r_k \).

The root mean square error of \( \hat{\phi}_j \), \( 1 \leq j \leq p \), defined by

\[
\sqrt{\text{E} (\hat{\phi}_j - \phi_j)^2} ,
\]

is of order \( \frac{1}{\sqrt{n}} \), and we can therefore approximate \( \hat{r}_k \) by a first order Taylor expansion about \( \hat{\phi} = \phi \); that is,

\[
\hat{r}_k = r_k + \sum_{j=1}^{p} (\hat{\phi}_j - \phi_j) \delta_{jk} + o(\frac{1}{n}) \tag{2.12}
\]

where

\[
\delta_{jk} = \frac{\partial \hat{r}_k}{\partial \phi_j} \bigg|_{\hat{\phi} = \phi} .
\]

Now

\[
\frac{\partial}{\partial \phi_j} \left[ \sum \hat{a}_t^2 \right] = 0 \text{ at } \hat{\phi} = \phi ,
\]

so that

\[
\delta_{jk} = -\left[ \sum \hat{a}_t^2 \right]^{-1} \frac{\partial \hat{c}_k}{\partial \phi_j} \bigg|_{\hat{\phi} = \phi} \tag{2.13}
\]
where

\[ c_k = \sum \hat{\dot{a}_t} \hat{a}_{t-k} \]

\[ = \sum (y_t - \hat{\phi}_1 y_{t-1} - \cdots - \hat{\phi}_p y_{t-p})(y_{t-k} - \hat{\phi}_1 y_{t-k-1} - \cdots - \hat{\phi}_p y_{t-k-p}) \]

\[ = \sum_{t} \sum_{i=0}^{p} \sum_{j=0}^{p} \hat{\phi}_i \hat{\phi}_j y_{t-i} y_{t-j} \]

(2.14)

where in (2.14) and below, \( \hat{\phi}_0 = \hat{\dot{\phi}}_0 = -1 \). From (2.13) and (2.14) it follows that

\[ \hat{\delta}_{jk} = -\left[ \frac{\sum y_t^2}{\sum \hat{\phi}_i^2} \right] \sum_{i=0}^{p} \hat{\phi}_i \frac{y_{k-i+j} y_{k+i-j}}{(r_{k-i+j} + r_{k+i-j})} \]

\[ = -\frac{\sum_{i=0}^{p} \sum_{j=0}^{p} \hat{\phi}_i \hat{\phi}_j y_{t-j}}{\sum_{i=0}^{p} \sum_{j=0}^{p} \hat{\phi}_i \hat{\phi}_j r_{i-j}} \]

(2.15)

where

\[ y_{r_{u}} = \frac{\sum y_t y_{t-u}}{\sum y_t^2} \]

Let \( \delta_{jk} \) be the result of replacing \( \hat{\phi} \)'s and \( r \)'s in (2.15) by \( \phi \)'s and \( \rho \)'s [the theoretical parameters and autocorrelations of the autoregressive process \( \{y_t\} \)]. That is,

\[ \delta_{jk} = \frac{\sum_{i=0}^{p} \phi_i (\rho_{k-i+j} + \rho_{k+i-j})}{-\sum_{i=0}^{p} \sum_{j=0}^{p} \phi_i \phi_j \rho_{i-j}} \]

(2.16)
Now from Bartlett's formula \([2], \text{eq. (7)}\) we have

\[ r_k = \rho_k + o\left(\frac{1}{\sqrt{n}}\right), \]

and as in the discussion preceding (2.12),

\[ \hat{\phi}_j = \phi_j + o\left(\frac{1}{\sqrt{n}}\right). \]

Since the denominator of (2.15) is of order 1 it follows that

\[ \hat{\delta}_{jk} = \delta_{jk} + o\left(\frac{1}{\sqrt{n}}\right). \quad (2.17) \]

Therefore by substituting \(\delta_{jk}\) for \(\hat{\delta}_{jk}\) in (2.12) the accuracy of the latter expression is preserved; that is,

\[ \hat{r}_k = r_k + \sum_{j=1}^{p} (\phi_j - \hat{\phi}_j) \delta_{jk} + o\left(\frac{1}{n}\right). \quad (2.18) \]

**Equality between the derivatives and \(\psi\)-weights.**

By making use of the recursive relation which is satisfied by the autocorrelations of an autoregressive process, namely

\[ \rho_v = \phi_1 \rho_{v-1} + \cdots + \phi_p \rho_{v-p}, \quad v \geq 1, \]

or

\[ \phi(B) \rho_v = 0, \quad v \geq 1, \quad (2.19) \]

expression (2.16) can be simplified to yield

\[ \delta_{jk} = \frac{\sum_{i=0}^{p} \phi_i \rho_{k-j+i}}{\sum_{i=0}^{p} \phi_i \rho_i}. \quad (2.20) \]

Thus \(\delta_{jk}\) depends only on \((k-j)\), and we therefore write \(\delta_{k-j} = \delta_{jk}\). Then
\( \delta_0 = 1, \) \hspace{1cm} (2.21)

and since

\[
\sum_{i=0}^{P} \phi_i \rho_{v+i} = \sum_{i=0}^{P} \phi_i \rho_{-v-i} = -\phi(B) \phi_v,
\]

it follows from (2.19) that

\( \delta_v = 0, \quad v < 0, \)

that is,

\( \delta_{jk} = 0, \quad k < j. \) \hspace{1cm} (2.22)

For \( v \geq 1, \) i.e. \( k > j, \) we therefore have

\[
\phi(B) \delta_v = \frac{\sum_{i=0}^{P} \phi_i [\phi(B) \rho_{v+i}]}{\sum_{i=0}^{P} \phi_i \rho_i} = 0 \hspace{1cm} (2.23)
\]

Combining (2.23) with the corresponding result (2.4) for \( \psi_v, \) we have

\( \delta_v = \psi_v, \) that is, \( \delta_{jk} = \psi_{k-j}. \) \hspace{1cm} (2.24)

**Representation of \( \hat{r} \) as a linear transformation of \( r. \)**

We can now establish a relationship between the residual autocorrelations \( \hat{r} \) and the white noise autocorrelations \( r. \) Let
\[
X = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\psi_1 & 1 & 0 & & \\
\psi_2 & \psi_1 & 1 & \cdots & \\
\psi_3 & \psi_2 & \psi_1 & & 0 \\
& & & \ddots & \psi_1 \\
\psi_{m-1} & \psi_{m-2} & \psi_{m-3} & \cdots & \psi_{m-p}
\end{bmatrix}
\]

\[
= [x_1 | x_2 | x_3 | \cdots | x_p].
\]

Then from (2.24) we can write (2.18) in vector form as

\[
\hat{r} = r + X(\hat{\phi} - \hat{\phi})
\]

(2.26)

where from (2.11),

\[
\hat{r}' X = 0.
\]

(2.27)

If we now multiply (2.26) on both sides by

\[
Q = X(X' X)^{-1} X'
\]

we obtain

\[
0 = Q r + Q X(\hat{\phi} - \hat{\phi})
\]

\[
= Q r + X(\hat{\phi} - \hat{\phi})
\]

\[
= Q r + \hat{r} - r
\]

and thus finally

\[
\hat{r} = (I - Q) r.
\]

(2.28)
It is known [1] that \( \mathbf{r} \) is very nearly normal for \( n \) moderately large. The vector of residuals is thus approximately a linear transformation of a multi-normal variable and is therefore itself normally distributed. Specifically

\[
\mathbf{r} \sim N(0, \frac{1}{n} \mathbf{I}),
\]

and hence

\[
\hat{\mathbf{r}} \sim N(0, \frac{1}{n} [\mathbf{I} - \mathbf{Q}]). \tag{2.29}
\]

We recall that this approximation works best when \( \hat{\mathbf{r}} \) is of dimension \( m \) as in the discussion following (2.11). The matrix \( \mathbf{I} - \mathbf{Q} \) is then idempotent of rank \( m - p \), so that the distribution of \( \hat{\mathbf{r}} \) has a \( p \)-dimensional singularity.

2.2. Further consideration of the covariance structure of the \( \hat{\mathbf{r}} \)'s.

It is illuminating to examine in greater detail the covariance matrix of \( \hat{\mathbf{r}} \), or equivalently the matrix \( \mathbf{Q} \). The latter matrix is idempotent of rank \( p \), and its non-null latent vectors are the columns of \( \mathbf{X} \). Also,

\[
\mathbf{X}' \mathbf{X} = \begin{bmatrix}
\sum \psi_j^2 & \sum \psi_j \psi_{j-1} & \cdots & \sum \psi_j \psi_{j-p+1} \\
\sum \psi_j \psi_{j-1} & \sum \psi_j^2 & \cdots & \sum \psi_j \psi_{j-p+2} \\
\vdots & \vdots & \ddots & \vdots \\
\sum \psi_j \psi_{j-p+1} & \sum \psi_j \psi_{j-p+2} & \cdots & \sum \psi_j^2 \\
\end{bmatrix}
\]

\[
= \begin{pmatrix}
\frac{\sigma^2_y}{\sigma^2_a} \\
\rho_1 & \frac{1}{\sigma^2_a} \\
\rho_2 & \frac{1}{\sigma^2_a} & \ddots \\
\rho_{k-1} & \rho_{k-2} & \cdots & \frac{1}{\sigma^2_a} \\
\end{pmatrix}. \tag{2.30}
\]

which when multiplied by \( \sigma^2_a \) is the autocovariance matrix of the process
itself. Let $c_{ij}^{th}$ be the $(ij)^{th}$ element of $(X'X)^{-1}$, and similarly $q_{ij}$ for $Q$.

If $\xi_j = (\psi_{j-1}, \ldots, \psi_{j-p})$ denotes the $j^{th}$ row of $X$, then

$$q_{ij} = \xi_i' (X'X)^{-1} \xi_j$$

$$= \sum_{k=1}^{p} \sum_{\ell=1}^{p} \psi_{i-k} c_{k\ell} \psi_{j-\ell}$$

$$= (-n) \text{cov} [\hat{r}_i, \hat{r}_j] \text{ if } i \neq j.$$ (2.31)

Since the elements of each column of $X$ satisfy the recursive relation (2.4), we have

$$\phi(B) \xi_j = 0$$

and hence

$$\phi(B) q_{ij} = 0,$$ (2.32)

where in (2.32) $B$ can operate either on $i$ or on $j$. This establishes an interesting recursive structure in the residual autocorrelation covariance matrix $\frac{1}{n}(I - Q)$ and provides an important clue as to how rapidly the covariances die out and the variances approach 1. Also, because of this property the entire covariance matrix is determined by specifying the elements

$$q_{11} \quad q_{12} \ldots \quad q_{1p}$$

$$q_{22} \ldots \quad q_{2p}$$

$$\ldots$$

$$q_{pp}$$

of $Q$, (2.33)
which are readily obtained by inverting the \( X'X \) matrix (2.30).

Covariance matrix of \( \hat{\tau} \) for \(^{1}\text{st}\) and \(^{2}\text{nd}\) order processes.

Consider, for example, the first order autoregressive process

\[
y_t = (1 - \phi B)^{-1} a_t
\]

\[
= (1 + \phi B + \phi^2 B^2 + \ldots) a_t .
\]

(2.34)

For this process

\[
\psi_j = \phi^j , (X'X)^{-1} = 1 - \phi^2 .
\]

From (2.31) the \((i,j)^\text{th}\) element of \( Q \) is therefore \( \phi^{i+j-2} (1 - \phi^2) \), so that approximately the residual autocorrelation covariance matrix is

\[
\begin{bmatrix}
\phi^2 & -\phi + \phi^3 & -\phi + \phi^4 & -\phi + \phi^5 & \ldots \\
-\phi + \phi^3 & 1 - \phi + \phi^4 & -\phi + \phi^5 & -\phi + \phi^6 & \ldots \\
-\phi + \phi^4 & -\phi + \phi^5 & 1 - \phi + \phi^6 & -\phi + \phi^7 & \ldots \\
-\phi + \phi^5 & -\phi + \phi^6 & -\phi + \phi^7 & 1 - \phi + \phi^8 & \ldots \\
& & & & \\
& & & & \\
& & & & \\
\end{bmatrix}
\]

(2.35)

For the second order process

\[
y_t = (1 - \phi_1 B - \phi_2 B^2)^{-1} a_t
\]

\[
= (1 + \psi_1 B + \psi_2 B^2 + \ldots) a_t
\]

(2.36)

we have
\[
X = \begin{bmatrix}
1 & 0 \\
\psi_1 & 1 \\
\psi_2 & \psi_1 \\
\vdots & \vdots \\
\end{bmatrix}, \quad (X' X)^{-1} = \frac{\sigma_y^2}{\sigma_y^2 (1-\rho_1^2)} \begin{bmatrix}
1 & -\rho_1 \\
-\rho_1 & 1 \\
\end{bmatrix}
\]

and

\[
\sigma_y^2 = \frac{(1 - \phi_2) \sigma_a^2}{(1+\phi_2)(1-\phi_2)^2 - \phi_1^2} .
\]

Thus

\[
q_{11} = 1 - \phi_2^2 , \quad q_{12} = -\phi_1 \phi_2 (1 + \phi_2),
\]

\[
q_{22} = 1 - \phi_2^2 - \phi_1^2 (1 + \phi_2)^2
\]

from which \( Q \) and \( \hat{\Sigma} = \frac{1}{n}(I - Q) \) may be determined using (2.32). In particular,

\[
V(\hat{r}_1) = \frac{1}{n} \cdot \phi_2^2 ,
\]

\[
V(\hat{r}_2) = \frac{1}{n} [\phi_2^2 + \phi_1^2 (1 + \phi_2)^2] ,
\]

\[
\vdots
\]

\[
V(\hat{r}_k) = \frac{1}{n} [1 - \phi_1 q_{k,k-1} - \phi_2 q_{k,k-2}] .
\]

From these examples we can see a general pattern emerging. From (2.33) the first \( p \) variances and corresponding covariances will be heavily dependent
on the parameters $\phi_1, \ldots, \phi_p$ and in general can depart sharply from the corresponding values for white noise autocorrelations, whereas for $k \geq p + 1$ a "1" is introduced into the expression for variances [as in (2.37)] and the recursion (2.32) ensures that as $k$ increases the $\hat{r}_k$ behave increasingly like the corresponding $r_k$ with respect to both their variances and covariances.

2.3. The distribution of $n \sum_{l=1}^{m} \hat{r}_{k_l}^2$.

We have remarked earlier that if the fitted model is appropriate and the parameters $\phi$ are exactly known, then the calculated $a_t$'s would be uncorrelated normal deviates, their serial correlations $r_k$ would be approximately $N(0, \frac{1}{n} I)$, and thus $n \sum_{l=1}^{m} \hat{r}_{k_l}^2$ would possess a $\chi^2$ distribution with $m$ degrees of freedom. We now see that if $m$ is taken sufficiently large so that the elements after the $m^{th}$ in the latent vectors of $Q$ are essentially zero, then we should expect that to the order of approximation we are here employing, the statistic

$$n \sum_{l=1}^{m} \hat{r}_{k_l}^2,$$

obtained when estimates $\hat{\phi}$ are substituted for the true parameters $\phi$ in the model, will still be distributed as $\chi^2$, only now with $m - p$ rather than $m$ degrees of freedom. This result is of considerable practical interest because it suggests that an overall test of the type discussed by Box and Jenkins can in fact be justified when suitable modifications coming from a more careful analysis are applied. Later we consider in more detail the use of this test, along with procedures based on individual $r$'s, in diagnostic checking.
3. Monte-Carlo Experiment

We have made certain approximations in deriving the distribution theory of the residual autocorrelations, and it is of interest to obtain empirical variances and correlations of \( \hat{r} \) through repeated sampling and to compare them with the corresponding theoretical values. This was done for the first order AR process for \( \phi = 0, \pm 1, \pm 3, \pm 5, \pm 7, \pm 9 \), although we will give greatest consideration to the values \( \phi = .3, .5, .7, .9 \).

Thus for given \( \phi \), \( s = 50 \) sets of \( n = 200 \) random normal deviates were generated on the computer using a method described in [5] (separate aggregates of deviates were obtained for each parameter value). For the \( j \)th set a series \( \{y_t^{(j)}\} \) was generated using formula (2.36), \( \hat{r}(j) \) was estimated, \( \{\hat{a}_t^{(j)}\} \) determined, and the quantities

\[
\hat{r}_k^{(j)} = \frac{\sum \hat{a}_n^{(j)} \hat{a}_{n-k}^{(j)}}{\sum \hat{a}_n^{(j)}^2}
\]

computed for \( 1 \leq k \leq m = 20, 1 \leq j \leq s = 50 \). This yielded sample variances and covariances

\[
C_{kk} = \frac{1}{50} \sum_{j=1}^{50} \hat{r}_k^{(j)} \hat{r}_k^{(j)}
\]

and sample correlations

\[
R_{kk} = \frac{C_{kk}}{\sqrt{C_{kk} C_{kk}}}
\]

The theoretical [based on (2.29)] and empirical [from Monte-Carlo sampling] variances of \( \hat{r}_1, \ldots, \hat{r}_{20} \) are shown in table 1, and their correlations in tables 2, 3, 4, and 5. It is seen that whenever the theoretical correlations depart markedly from the zeros of the identity correlation matrix for white noise autocorrelations, so do the Monte-Carlo
TABLE 1. Ratio of theoretical [as in (2.35)] and empirical [from Monte-Carlo sampling] residual autocorrelation variances to the corresponding value of 1/n for white noise autocorrelations, for 1st order AR process.

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TABLE 2(a). THEORETICAL CORRELATION MATRIX, $\phi_1 = .3$

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estimates and in roughly the same manner; and whenever the theoretical
correlations are close to zero, the Monte-Carlo values tend to be distributed
about zero. Analogous remarks can also be made for the theoretical and
empirical variances.

Since \( V(\hat{r}_1) \) and to a lesser extent \( V(\hat{r}_2) \) depart the most from the
responding values of \( \frac{1}{n} \) for \( r_1, r_2 \), a more detailed comparison between
theoretical and empirical values of these quantities is made in figure 1,
for \( \phi = 0, \pm 1, \pm 3, \pm 5, \pm 7, \pm 9 \). There is very good agreement, at least
for \( V(\hat{r}_1) \).

There are several additional comparisons which can be made based on certain
functions of the \( \hat{r}'s \). Thus we have seen that

\[
\hat{\chi} = \sum \phi^{k-1} \hat{r}_k = 0
\]

and in the course of our derivations we have had to make the approximation

\[
\hat{\chi} = \sum \phi^{k-1} \hat{r}_k = 0. \tag{3.1}
\]

Some indication of the validity of this approximation is gained by examining
the actual values of \( \hat{\chi} \) from the sampling experiment, which were found to be
distributed about zero with a variance of about one one-hundredth that which
would have been expected from the same linear form in white noise autocorrelations.

Of considerable importance because of its role in diagnostic checking
is an examination of the quantity

\[
n \sum_{k=1}^{m} \hat{r}_k^2 = 200 \sum_{k=1}^{20} \hat{r}_k^2 \tag{3.2}
\]

which as in (2.37) should possess a \( \chi^2 \) - distribution with \( v = m-1 = 19 \)
degrees of freedom. Such a distribution has a mean and variance of 19 and
38, respectively, and the corresponding empirical means and variances are
shown in table 6. It is apparent from examining these results that the
FIGURE 1. Theoretical (line) and empirical (dots) variances of $r_1$

FIGURE 2. Theoretical (line) and empirical (dots) variances of $r_2$
TABLE 6. Empirical means and variances of

\[ S = 200 \sum_{k=1}^{20} r_k^2 \]

\[
\text{Mean} = \overline{S} = \frac{1}{50} \sum_{j=1}^{50} s(j) ; \quad \text{Variance} = \frac{1}{49} \sum_{j=1}^{50} (s_j - \overline{S})^2
\]

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Grand mean = 18.1 with std. error of \( \sqrt{\frac{2\nu}{N}} = \sqrt{\frac{38}{500}} = .28 \)

Pooled variance = 39.4 with std. error of \( 2\nu \sqrt{\frac{2+\nu}{N}} = 38 \sqrt{\frac{12}{500}} = 2.76 \)
over-all mean of 18.1 is significantly different from the theoretical value of 19 characteristic of a \(\chi^2_{19} \) variate, which at first glance may be puzzling. However the quantity \( n \sum_{k=1}^{m} \hat{r}_k^2 \) possesses a \(\chi^2_{m-p} \) distribution only insofar as the white noise autocorrelations \( \hat{r}_k = (r_1', \ldots, r_m') \) have a common variance of \( \frac{1}{n} \), and from (1.4) it is seen that this approximation overestimates the true variance of a given \( r_k \) by a factor of \( \frac{n+2}{n-k} \). In particular, for \( n = 200 \), \( m = 20 \), and a typical value of \( k = 10 \), the actual variance \( V(r_k) \) is \( \frac{190}{202} \approx 94\% \) of the \( \frac{1}{n} \)-approximation. Since the residual autocorrelations \( \hat{r}_k \) are by (2.28) a linear transformation of \( r_k \), it is reasonable to expect that a comparable depression of the variances of \( \{ \hat{r}_k \} \) would occur, and this would account for the discrepancy between the theoretical and empirical means of the statistic \( \frac{200}{n} \sum_{k=1}^{m} \hat{r}_k^2 \) encountered above. [This phenomenon would also explain the tendency for the empirical variances themselves (table 1) to take on values averaging about 5\% lower than those based on the matrix \( \frac{1}{n}(I - Q) \) of (2.29)].

4. Use of residual autocorrelations in diagnostic checking.

We have obtained the large sample distribution of the residual autocorrelations \( \hat{r}_k \) from fitting the correct model to a time series, and we have discussed the ways in which this distribution departs significantly from that of white noise autocorrelations \( r_k \). It is desirable now to consider the practical implications of these results in examining the adequacy of fit of a model.

First of all it appears that even though the \( \hat{r}_k \)'s have a variance/covariance matrix which can differ very considerably from that of the \( r_k \)'s, the statistic \( \frac{200}{n} \sum_{k=1}^{m} r_k^2 \) will (since this matrix is idempotent) still possess a \( \chi^2 \)-distribution, only now with \( m - p \) rather than \( m \) degrees of freedom. Thus the overall \( \chi^2 \)-test discussed in section 1 may be justified to the same degree of
approximation as before when the number of degrees of freedom is appropriately
modified.

However, regarding the "quality - control - chart" procedure, that is the
comparison of the \( \{ \hat{r}_k \} \) with their standard errors, some modification is clearly
needed.

Figure 3 shows the straight-line standard error bands of width \( \frac{1}{\sqrt{n}} \)
associated with any set of white noise autocorrelations \( \{ r_k \} \). These stand in
marked contrast to the corresponding bands for the residual autocorrelations
\( \{ \hat{r}_k \} \), derived from their covariance matrix \( \frac{1}{n} (I - Q) \) and shown in figure 4 for
selected 1st and 2nd order AR processes. Since it is primarily the \( \hat{r} \)'s of
small lags that are most useful in revealing model inadequacies, we see that
the consequence of treating \( \hat{r} \)'s as \( r \)'s in the diagnostic checking procedure can
be a serious underestimation of significance, that is, a failure to detect lack
of fit in the model when it exists. Of course, if the model would have been
judged inadequate anyway, our conviction in this regard is now strengthened.

Suppose, for example, that we identify a series of length 200 as first
order autoregressive and after fitting \( \hat{\phi} = .5 \). Suppose also that \( \hat{r}_1 = .10 \).
Now the standard error of \( r_1 \) for white noise is \( \frac{1}{\sqrt{n}} = .07 \), so that \( \hat{r}_1 \) is well
within the limits in figure 3. Therefore if we erroneously regarded these as
limits on \( \hat{r}_1 \) we would probably conclude that this model was adequate. However
if the true process actually were 1st order autoregressive (say with \( \phi = .5 \)),
the standard error of \( \hat{r}_1 \) would be \( |\phi|/\sqrt{n} = .035 \); and since the observed
\( \hat{r}_1 = .10 \) is almost three times this value, we should be very suspicious of the
adequacy of this fit.

The situation is further complicated by the existence of rather high
correlations between the \( \hat{r} \)'s, especially those of small lags. This can
clearly be seen by examining tables 2, 3, 4, 5(a) for the AR (1) process.
The most serious correlation here is
FIGURE 3. Standard error limits for white noise autocorrelations $r_k$

FIGURE 4. Standard error limits for residual autocorrelations $\hat{r}_k$.

(a) AR(1), $\phi = .3$

(b) AR(1), $\phi = .7$

(c) AR(2), $\phi_1 = .5$, $\phi_2 = .25$

(d) AR(2), $\phi_1 = 1.0$, $\phi_2 = -.75$
\[ \rho[\hat{r}_1, \hat{r}_2] = \frac{-\phi}{\sqrt{1 - \phi^2}} \frac{1 - \phi^2}{\sqrt{1 - \phi^2} + \phi^4} \]

which approaches \(-1\) as \(\phi \to 0\) and is still as large as \(-.6\) for \(\phi = .7\). (In general for small values of \(|\phi|\) there is a high correlation between \(\hat{r}_1, \hat{r}_2\) and a quick return to zero for correlations between higher-lagged \(r\)'s, while for larger parameter values the nonzero correlations, while still most serious for small lags, are more distributed throughout the correlation matrix.)

Correlation among the \(\hat{r}\)'s is even more prevalent in second and higher-order processes, where in general (as for variances) those involving lags up to \(k = p\) can be particularly serious. From then on their magnitude is controlled by the recursive relationship (2.32). In particular, the closer \(\phi\) is to the boundary of the stationarity region, the slower will be the dying out of \(\text{cov}(\hat{r}_k, \hat{r}_k)\) or \(\rho(\hat{r}_k, \hat{r}_k)\), although often in these situations the less serious will the initial correlations \(\rho(\hat{r}_1, \hat{r}_2), \rho(\hat{r}_2, \hat{r}_3), \rho(\hat{r}_1, \hat{r}_3), \text{etc.}\) tend to be.

We have thus seen that the departure of the distribution of the residual autocorrelations \(\hat{r}\) from that of white noise autocorrelations \(r\) is serious enough to warrant some modifications in their use in diagnostic checking. The residual autocorrelation function however remains a powerful tool for this purpose, and since the principal shortcoming of the practice of regarding the \(\hat{r}\)'s as \(r\)'s was found to be an underestimation of significant model inadequacies, it follows that to the extent that we can use our knowledge regarding variances and intracorrelations of the \(\hat{r}\)'s, their usefulness to us as a diagnostic tool is increased.

5. Distribution of Residual Autocorrelations For the Moving Average and Mixed Processes.

In obtaining the distribution of \(\hat{r} = (\hat{r}_1, \ldots, \hat{r}_m)'\) for the pure auto-
regressive process (2.1) in the previous section, considerable use was made of the recursive relation

\[ \phi(B) \rho_k = 0 \]

which is not satisfied by moving average models, or more generally by mixed models of the form

\[ \phi(B) z_t = \theta(B) a_t \quad (5.1) \]

where

\[ \phi(B) = (1 - \phi_1 B - \ldots - \phi_p B^p), \]

\[ \theta(B) = (1 - \theta_1 B - \ldots - \theta_q B^q), \]

the \{a_t\} are as in (2.1), and the roots of both polynomials \(\phi(B)\) and \(\theta(B)\) have modulus greater than 1.

It is fortunate, therefore, that these models have in common with the pure AR models (2.1) an important property because of which the distribution of their residual autocorrelations can be found as an immediate consequence of the autoregressive solution (2.29). This property is that if two time series

\[ \phi(B) z_t = \theta(B) a_t \quad (5.1) \]

and

\[ \pi(B) x_t = a_t \quad (5.2) \]

are both generated from the same set of deviates \{a_t\}, and moreover if

\[ \pi(B) = \phi(B) \theta(B) \quad (5.3) \]

then when these models are fitted their residuals, and hence also their
residual autocorrelations, will be identically the same [to the extent that the moving average part of (5.1) is linear in \( \hat{\theta} \) in a region between \( \theta \) and \( \hat{\theta} \), which to \( o(1/\sqrt{n}) \) is a justifiable assumption]. Therefore if a mixed model of order \((p, d, q)\) is correctly identified and fitted, its residual autocorrelations will be distributed as though the model had been of order \((p + q, d, 0)\) with the relations between the two sets of parameters given by (5.3).

5.1. Equality of residuals in mixed and pure AR models.

Let \( z_t \) and \( x_t \) be as in (5.1) and (5.2); (5.3) then implies

\[
z_t = \theta^2(B) x_t.
\]

(5.4)

As in (2.5), define

\[
\hat{a}_t^{\text{AR}} = a_t^{\text{AR}(\hat{\pi})} = \pi(B) x_t
\]

\[
= (1 - \pi_1 B - \ldots - \pi_{p+q} B^{p+q}) x_t
\]

(5.5)

and now also

\[
\hat{a}_t^{*} = a_t^{\ast}(\hat{\phi}, \hat{\theta}) = \phi(B) \theta^{-1}(B) z_t
\]

\[
= (1 - \phi_1 B - \ldots - \phi_{p} B^{p})(1 - \theta_1 B - \ldots - \theta_{q} B^{q})^{-1} z_t
\]

(5.6)

We will expand these quantities about the true parameter values and go through a regression in each case which is analogous to writing

\[
y = X \beta + \epsilon
\]

as

\[
\hat{\epsilon} = y - \hat{\mu} = X (\hat{\beta} - \beta) + \epsilon.
\]
The equality of the residuals in the two cases depends heavily on the fact that the derivatives in each expansion involve the same autoregressive variable $x_t$.

Thus

$$
\frac{\partial \hat{a}_t}{\partial \pi_j} = - x_{t-j}, \ 1 \leq j \leq p + q,
$$

irrespective of $\pi$;

$$
\frac{\partial \hat{a}_t}{\partial \phi_j} = - \theta^{-1} (B) z_{t-j}, \ 1 \leq j \leq p
$$

$$
= - \theta(B) x_{t-j} \text{ at } \theta = \theta^*,
$$

irrespective of $\phi$; and

$$
\frac{\partial \hat{a}_t}{\partial \theta_j} = \phi(B) \theta^{-2}(B) z_{t-j}, \ 1 \leq j \leq q
$$

$$
= \phi(B) x_{t-j} \text{ at } \theta = \theta^*.
$$

Then

$$
\hat{a}_t^{AR} = a_t^{AR} + \sum_{j=1}^{p+q} (\pi_j - \hat{\pi}_j) x_{t-j}
$$

(5.7)

and

$$
\hat{a}_t^{*} = a_t^{*} + \sum_{i=1}^{p} (\phi_i - \hat{\phi}_i) \theta(B) x_{t-i}
$$

$$
- \sum_{j=1}^{q} (\theta_j - \hat{\theta}_j) \phi(B) x_{t-j}
$$

(5.8)
\[ a_t^* = \sum_{i=1}^{p} \left( \phi_i - \dot{\phi}_i \right) x_{t-i} - \sum_{j=1}^{q} \left( \theta_j - \dot{\theta}_j \right) x_{t-j} \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{q} \left( \phi_i - \dot{\phi}_i \right) \theta_j x_{t-i-j} \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{q} \left( \theta_j - \dot{\theta}_j \right) \phi_i x_{t-i-j} \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{q} \left[ \phi_i \left( \theta_j - \dot{\theta}_j \right) - \theta_j \left( \phi_i - \dot{\phi}_i \right) \right] x_{t-i-j} \]
\[ = a_t^* + \sum_{i=1}^{p} \left( \phi_i - \dot{\phi}_i \right) x_{t-i} - \sum_{j=1}^{q} \left( \theta_j - \dot{\theta}_j \right) x_{t-j} \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{q} \left[ \phi_i \left( \theta_j - \dot{\theta}_j \right) - \theta_j \left( \phi_i - \dot{\phi}_i \right) \right] x_{t-i-j} \]
\[ = a_t^* + \sum_{j=1}^{p+q} \left( \beta_j - \dot{\beta}_j \right) x_{t-j}. \] (5.9)

Thus if \( \beta = (\beta_1, \ldots, \beta_{p+q})' \) and \( \lambda = \begin{bmatrix} \phi \\ \theta \end{bmatrix} \), we see that
\[ \beta = A \lambda, \] (5.10)

where \( A \) is a \((p + q)\) - square matrix whose elements involve \( \lambda \) but not the true parameter values \( \lambda \). For example, if \( p = q = 1 \) we would have
\[ \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -\theta & \phi \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix}. \] (5.11)

Equations (5.7) and (5.9) can be written as
\[ a_{\text{AR}}^* = a + X (\pi - \hat{\pi}) \]  \hspace{1cm} (5.12)

\[ a^* = a + X (\beta - \hat{\beta}) \]  \hspace{1cm} (5.13)

where we have made use of the fact that, at \( \pi = \bar{\pi}, \beta = \bar{\beta}, \) and \( \phi = \bar{\phi}, \)

\[ a_{\text{AR}} = a^* = a_t. \]  \hspace{1cm} (5.14)

In (5.12) the sum of squares

\[ a' a = \sum a_t^2 = \sum [a_{\text{AR}} (\pi)]^2 \]

is minimized as a function of \( \pi \) when

\[ \pi - \hat{\pi} = \tilde{\pi} - \hat{\pi} = (X' X)^{-1} X' \hat{a}_{\text{AR}}, \]  \hspace{1cm} (5.15)

while in (5.13) if we write

\[ a^* = a + X [A (\lambda - \hat{\lambda})] \]

\[ = a + Z (\lambda - \hat{\lambda}) \]

then the sum of squares

\[ a' a = \sum a_t^2 = \sum [a^* (\lambda)]^2 \]

is minimized as a function of \( \lambda \) when

\[ \lambda - \hat{\lambda} = \tilde{\lambda} - \hat{\lambda} = (Z' Z)^{-1} Z' \hat{a}^* \]

\[ = A^{-1} (\hat{\beta} - \hat{\beta}); \]

that is,

\[ \hat{\beta} - \hat{\beta} = (X' X)^{-1} X' \hat{a}^*. \]  \hspace{1cm} (5.16)
Then by setting \( \hat{\pi} = \pi \) in (5.15) and (5.16) we have the important equality

\[
\hat{\pi} - \pi = (X'X)^{-1}X'a = \hat{\beta} - \beta. \tag{5.17}
\]

By setting ".-" in (5.12) and (5.13) we have finally from (5.17),

\[
\hat{a}^{AR} = a + X(\pi - \hat{\pi}) = a + X(\beta - \hat{\beta}) = \hat{a}^\ast. \tag{5.18}
\]

as we set out to show.

5.2 Monte Carlo experiment.

The equality (5.18) between the residuals from the autoregressive and mixed models depends on the accuracy of the expansion (5.8), that is, on the extent of linearity in the moving average model between the true and estimated values \( \pi \) and \( \hat{\pi} \). It is therefore worthwhile to confirm this model-duality by generating and fitting pairs of series of the form (5.1) and (5.2) and comparing their residuals, or more to our purpose, their residual autocorrelations. This was done for \( p + q = 1 \) and \( p + q = 2 \) for series of length 200, and the results for selected parameter values are shown in tables 7 and 8.

It is seen that the residual autocorrelations \( \hat{r}_k^{AR} \) and \( \hat{r}_k^{MA} \) [and \( \hat{r}_k^\ast \) for the mixed (1, 0, 1) model] are equal or nearly equal to the second decimal place.

A sampling experiment of the type described in section 3 for the AR(1) process was also performed for the first order moving average process. The results were very similar, which is to be expected in view of (5.18).
TABLE 7. Residual autocorrelations from 1st order AR and MA time series generated from same white noise (n = 200).

<table>
<thead>
<tr>
<th>k</th>
<th>$\hat{\phi} = 0 = .1$</th>
<th>$\hat{\phi} = 0 = .5$</th>
<th>$\hat{\phi} = 0 = .9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{r}_k^{\text{AR}}$</td>
<td>$\hat{r}_k^{\text{MA}}$</td>
<td>$\hat{r}_k^{\text{AR}}$</td>
</tr>
<tr>
<td>1</td>
<td>-.029</td>
<td>-.010</td>
<td>.003</td>
</tr>
<tr>
<td>2</td>
<td>.164</td>
<td>.169</td>
<td>.044</td>
</tr>
<tr>
<td>3</td>
<td>.096</td>
<td>.099</td>
<td>-.098</td>
</tr>
<tr>
<td>4</td>
<td>-.050</td>
<td>-.049</td>
<td>.014</td>
</tr>
<tr>
<td>5</td>
<td>-.003</td>
<td>-.006</td>
<td>.057</td>
</tr>
<tr>
<td>6</td>
<td>-.143</td>
<td>-.144</td>
<td>.010</td>
</tr>
<tr>
<td>7</td>
<td>-.023</td>
<td>-.026</td>
<td>-.004</td>
</tr>
<tr>
<td>8</td>
<td>-.040</td>
<td>-.041</td>
<td>-.054</td>
</tr>
<tr>
<td>9</td>
<td>.010</td>
<td>.009</td>
<td>.052</td>
</tr>
<tr>
<td>10</td>
<td>-.049</td>
<td>-.049</td>
<td>-.065</td>
</tr>
<tr>
<td>11</td>
<td>.029</td>
<td>.025</td>
<td>-.071</td>
</tr>
<tr>
<td>12</td>
<td>-.130</td>
<td>-.131</td>
<td>-.081</td>
</tr>
<tr>
<td>13</td>
<td>-.018</td>
<td>-.021</td>
<td>-.021</td>
</tr>
<tr>
<td>14</td>
<td>-.065</td>
<td>-.065</td>
<td>.077</td>
</tr>
<tr>
<td>15</td>
<td>.012</td>
<td>.011</td>
<td>.129</td>
</tr>
<tr>
<td>16</td>
<td>-.008</td>
<td>-.007</td>
<td>-.032</td>
</tr>
<tr>
<td>17</td>
<td>.015</td>
<td>.015</td>
<td>.003</td>
</tr>
<tr>
<td>18</td>
<td>.022</td>
<td>.023</td>
<td>.036</td>
</tr>
<tr>
<td>19</td>
<td>.076</td>
<td>.076</td>
<td>.109</td>
</tr>
<tr>
<td>20</td>
<td>.006</td>
<td>.007</td>
<td>-.006</td>
</tr>
</tbody>
</table>

$\hat{\phi}$ or $\hat{\theta}$ += .159 .057 .543 .451 .922 .870
TABLE 8. Residual autocorrelations from 2nd order AR, 2nd order MA, and 2-parameter mixed series generated from same white noise (n = 200).

<table>
<thead>
<tr>
<th>k</th>
<th>( ^*r _AR )</th>
<th>( ^*r _AR )</th>
<th>( r _MA )</th>
<th>( ^*r _AR )</th>
<th>( ^*r _AR )</th>
<th>( r _MA )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.004</td>
<td>-0.002</td>
<td>-0.008</td>
<td>0.064</td>
<td>0.049</td>
<td>0.049</td>
</tr>
<tr>
<td>2</td>
<td>0.011</td>
<td>0.012</td>
<td>0.013</td>
<td>-0.127</td>
<td>-0.100</td>
<td>-0.097</td>
</tr>
<tr>
<td>3</td>
<td>-0.018</td>
<td>-0.018</td>
<td>-0.016</td>
<td>-0.035</td>
<td>-0.035</td>
<td>-0.054</td>
</tr>
<tr>
<td>4</td>
<td>0.037</td>
<td>0.036</td>
<td>0.038</td>
<td>0.086</td>
<td>0.103</td>
<td>0.105</td>
</tr>
<tr>
<td>5</td>
<td>-0.008</td>
<td>-0.008</td>
<td>-0.008</td>
<td>0.018</td>
<td>-0.003</td>
<td>-0.003</td>
</tr>
<tr>
<td>6</td>
<td>0.017</td>
<td>0.015</td>
<td>0.015</td>
<td>0.064</td>
<td>0.078</td>
<td>0.081</td>
</tr>
<tr>
<td>7</td>
<td>0.013</td>
<td>0.013</td>
<td>0.013</td>
<td>-0.092</td>
<td>-0.105</td>
<td>-0.107</td>
</tr>
<tr>
<td>8</td>
<td>-0.111</td>
<td>-0.112</td>
<td>-0.112</td>
<td>-0.024</td>
<td>-0.007</td>
<td>-0.008</td>
</tr>
<tr>
<td>9</td>
<td>0.022</td>
<td>0.022</td>
<td>0.023</td>
<td>-0.056</td>
<td>-0.075</td>
<td>-0.074</td>
</tr>
<tr>
<td>10</td>
<td>-0.109</td>
<td>-0.110</td>
<td>-0.110</td>
<td>-0.015</td>
<td>-0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>11</td>
<td>0.070</td>
<td>0.070</td>
<td>0.071</td>
<td>-0.074</td>
<td>-0.086</td>
<td>-0.089</td>
</tr>
<tr>
<td>12</td>
<td>-0.039</td>
<td>-0.039</td>
<td>-0.039</td>
<td>-0.098</td>
<td>-0.087</td>
<td>-0.085</td>
</tr>
<tr>
<td>13</td>
<td>0.025</td>
<td>0.026</td>
<td>0.026</td>
<td>-0.101</td>
<td>-0.112</td>
<td>-0.113</td>
</tr>
<tr>
<td>14</td>
<td>0.060</td>
<td>0.060</td>
<td>0.061</td>
<td>-0.018</td>
<td>-0.004</td>
<td>-0.008</td>
</tr>
<tr>
<td>15</td>
<td>0.000</td>
<td>-0.001</td>
<td>-0.002</td>
<td>-0.069</td>
<td>-0.076</td>
<td>-0.078</td>
</tr>
<tr>
<td>16</td>
<td>0.052</td>
<td>0.051</td>
<td>0.051</td>
<td>0.031</td>
<td>0.039</td>
<td>0.038</td>
</tr>
<tr>
<td>17</td>
<td>-0.011</td>
<td>-0.012</td>
<td>-0.013</td>
<td>-0.016</td>
<td>-0.027</td>
<td>-0.026</td>
</tr>
<tr>
<td>18</td>
<td>0.121</td>
<td>0.122</td>
<td>0.122</td>
<td>-0.062</td>
<td>-0.053</td>
<td>-0.052</td>
</tr>
<tr>
<td>19</td>
<td>-0.020</td>
<td>-0.018</td>
<td>-0.020</td>
<td>-0.012</td>
<td>-0.017</td>
<td>-0.018</td>
</tr>
<tr>
<td>20</td>
<td>0.083</td>
<td>0.081</td>
<td>0.080</td>
<td>0.060</td>
<td>0.065</td>
<td>0.064</td>
</tr>
</tbody>
</table>

Parameter estimates:
- \( \phi_1 = 1.37 \), \( \phi_2 = 1.23 \), \( \theta_1 = -0.38 \), \( \theta_2 = -0.48 \)
5.3. Conclusions.

We have shown above that to a close approximation the residuals from any moving average or mixed autoregressive - moving average process will be the same as those from a suitably chosen autoregressive process. We have further confirmed the adequacy of this approximation by empirical calculation. It follows from this that we need not consider separately these two classes of processes; more precisely,

1. We can immediately use the AR result to write down the variance/covariance matrix of \( \hat{r} \) for any AR-MA model

\[
\phi(B) y_t = \theta(B) a_t
\]

by considering the corresponding variance covariance matrix of \( \hat{r} \) from the pure AR process

\[
\pi(B) w_t = \theta(B) \phi(B) w_t = a_t.
\]

Also, since \( y_t = \nabla^d z_t \) can be the \( d \)th difference of the time series under consideration, the results apply to the general integrated autoregressive - moving average (IARMA) model of order \( (p, d, q) \) set forth at the beginning of this paper.

2. All considerations regarding the use of residual autocorrelations in diagnostic checking discussed in section 4 for the autoregressive model therefore apply equally to moving average and mixed models.

3. In particular it follows from the above that a "portmanteau" test for the adequacy of any IARMA process is obtained by referring

\[
n \sum_{k=1}^{m} \hat{r}^2_k
\]

to a \( \chi^2 \) distribution with \( \nu \) degrees of freedom, where

\[
\nu = m - p - q.
\]
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