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EXTENDING THE FREQUENCY RANGE OF SPECTRUM
ESTIMATES BY THE USE OF TWO DATA RECORDERS.

by

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§1. Background

Let \( \{X_t; t = ... -2, -1, 0, 1, 2, ... \} \) be a real-valued, weakly stationary, discrete stochastic process (or time series) with zero mean and covariance function:

\[
R(\nu) = E[X_t X_{t+\nu}] = R(-\nu). \tag{1.1}
\]

The power spectrum or spectral density \( f(\cdot) \) of the process is the Fourier transform of the covariance function, i.e.

\[
f(\omega) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} R(\nu) \cos \nu \omega \tag{1.2}
\]

\[
R(\nu) = \int_{-\pi}^{\pi} f(\omega) \cos \nu \omega \, d\omega. \tag{1.3}
\]

The process itself has a spectral representation:

\[
X_t = \int_{-\pi}^{\pi} e^{i t \omega} \, dZ(\omega) \tag{1.4}
\]

where \( Z(\cdot) \) is a complex-valued stochastic function with uncorrelated increments, and

\[
E[|dZ(\omega)|^2] = f(\omega) \, d\omega \tag{1.5}
\]

(Cramer (1940)). This expresses \( X_t \) as a superposition of harmonics with random amplitudes, the expectation of whose square is proportional to the spectral density. The fundamental importance of the spectrum in the analysis of a stationary stochastic process is therefore that the tendency of the process to oscillate with a frequency \( \omega \) is reflected directly in the value of \( f(\omega) \).

The classical form of estimate of the spectrum using a finite sample \( \{X_t; t = 1, 2, ..., T\} \) is

\[
f_T(\omega) = \frac{1}{2\pi} \sum_{\nu=-T+1}^{T-1} k_T(\nu) \, R_T(\nu) \cos \nu \omega \quad (-\pi < \omega < \pi) \tag{1.6}
\]

where \( R_T(\cdot) \) is the sample covariance function.
time series analyst has to use: because of the physical apparatus through which the information must pass before it is finally recorded, the resulting data is always "essentially discrete or band limited (i.e. frequency limited)". Thus in any practical situation, there is a minimum time interval $\Delta t$ at which data may usefully be recorded.

If the range of time $t$ is the whole real line, (1.2), (1.3) and (1.4) become

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(v) \cos v\omega \, dv$$  \hspace{1cm} (1.12)

$$R(v) = \int_{-\infty}^{\infty} f(\omega) \cos v\omega \, d\omega$$  \hspace{1cm} (1.13)

$$X_t = \int_{-\infty}^{\infty} e^{it\omega} \, dZ(\omega).$$  \hspace{1cm} (1.14)

In particular, $R(\cdot)$ is now a function defined on the real line. The frequency range over which $f(\cdot)$ is defined is also infinite, and correspondingly the range of oscillation-periods covered is $(\infty, 0)$ instead of $(\infty, 2)$.

It is shown in Blackman and Tukey (1958) that the spectrum $f^A(\cdot)$ of the discrete time series $\{X_t; t = \ldots, -2, -1, 0, 1, 2, \ldots\}$ derived from a continuous process $\{X_t; -\infty < t < \infty\}$ with spectrum $f(\cdot)$, by recording only its values at integer time points, is given by

$$f^A(\cdot) = f(\omega) + \sum_{j=1}^{\infty} \{f(2\pi j + \omega) + (2\pi j - \omega)\}. \hspace{1cm} (1.15)$$

$f^A(\cdot)$ is called an ALIASED spectrum, and Tukey's illustrative description is that $\{f(\omega); 0 < \omega < \infty\}$ is "folded" backwards and forwards into the interval $(0, \pi)$ and the appropriate contributions added in order to obtain $f^A(\cdot)$.

Because of (1.15), care must be exercised in interpreting spectrum estimates which are more correctly estimates of $f^A(\cdot)$ unless there are reasons for assuming $f(\omega)$ to be very small for $\omega > \pi$ (i.e. for oscillation-
periods less than 2 time-units). The need may therefore arise for estimates of the spectrum over a frequency range \((0,\pi)\) which corresponds to a wider oscillation-period range than the \((\infty, 2\Delta t)\) which is generally available from data recorded by a mechanism whose smallest practical sampling period is \(\Delta t\). The following three paragraphs present material which is then used in §5 to enable such estimates to be obtained and their variances studied.

§2. Asymptotically stationary time series

The notion of an asymptotically stationary time series was introduced by Parzen (1961b) where, because of the practical way as opposed to the theoretical way in which the spectral analysis of time series is performed, he defined the covariance function \(R(\cdot)\) differently from (1.1). He required only that the limits

\[
\lim_{T \to \infty} R_T(\nu) = \lim_{T \to \infty} \sum_{t=1}^{T-|\nu|} X_t X_{t+|\nu|}, \nu = 0, \pm 1, \pm 2, \ldots \quad (2.1)
\]

should exist as a function \(R(\nu)\) for almost all realizations. This function is then defined as the covariance function. If \(\{X_t\}\) is stationary and ergodic, the two definitions coincide. The limits (2.1) do however exist in many cases where \(\{X_t\}\) is not stationary. All processes for which the covariance function exists in this sense are said to possess a generalised harmonic analysis and to be asymptotically stationary.

A necessary and sufficient condition for ergodicity is

\[
\lim_{T \to \infty} \text{var}(R_T(\nu)) = 0, \nu = 0, \pm 1, \pm 2, \ldots \quad (2.2)
\]

This follows from a theorem in Parzen (1961b), and is equivalent to

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{\nu=1}^{T} R(\nu)^2 = 0. \quad (2.3)
\]

A consequence of ergodicity is the consistency in quadratic mean of the estimate \(R_T(\nu)\) of \(R(\nu)\), the latter being defined in either of the above manners.
An important way in which an asymptotically stationary time series arises is by the amplitude modulation of a stationary process. If \( \{X_t\} \) is stationary with zero mean (for convenience) and covariance function \( R(\cdot) \), and \( g(\cdot) \) is a non-random bounded function defined over the integers which itself possesses a generalised harmonic analysis in the sense that there is a function \( R_g(\cdot) \) such that

\[
R_g(v) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T-v} g(t) g(t+v)
\]

(2.4)

exists, then the process \( \{Y_t\} \) defined by

\[
Y_t = g(t) X_t
\]

(2.5)
is called an amplitude modulated version of \( \{X_t\} \). Following Parzen, we call \( g(\cdot) \) the amplitude modulating function. Since

\[
E[Y_t Y_{t+v}] = g(t) g(t+v) R(v)
\]

(2.6)
it is obvious that \( \{Y_t\} \) is not stationary, but it is asymptotically stationary with covariance function \( R_Y(\cdot) \) given by

\[
R_Y(v) = R_g(v) R(v).
\]

(2.7)

Now it is shown in Parzen (1961b) that if \( \{X_t\} \) is an ergodic normal process, then \( \{Y_t\} \) is also ergodic. Consequently, given the sample \( \{Y_t; t = 1, 2, \ldots, T\} \), a consistent in quadratic mean estimate of \( R_Y(v) \) is given by the sample covariance function

\[
R_T^Y(v) = \frac{1}{T} \sum_{t=1}^{T-v} Y_t Y_{t+v}.
\]

(2.8)

A consistent estimate of \( R(v) \) is then available, viz.,

\[
\hat{R}(v) = \frac{R_T^Y(v)}{R_g(v)}
\]

(2.9)
as long as \( \text{Rg}(\nu) \neq 0 \). Thus consistent estimates of \( f(\cdot) \) are available in the usual form:

\[
\hat{f}_T^*(\omega) = \frac{1}{2\pi} \sum_{\nu=-M_T}^{M_T} k(\frac{\nu}{N_T}) \hat{R}(\nu) \cos \nu \omega
\]  

(2.10)

§3. Missing Data

We now consider situations where, instead of having a complete finite sample of the usual form \( \{X_t; \ t = 1, 2, \ldots, T\} \), some of these \( T \) observations are unread. The situation covered by Jones (1962) and Parzen (1962) was that of periodically missing data, which is where the data is read in blocks of size \( \alpha \), separated by blocks of size \( \beta \). Data which might be read in this way include astronomical measurements which can only be read when certain stars or the moon are visible, or data concerning sporting activities which naturally cannot be read out of season.

Such data may be dealt with by defining an amplitude modulating function \( g(t) \) as taking the value 1 or 0 according as the data is or is not read at the time \( t \) and recording zeros at the time points \( t \) for which \( g(t) = 0 \). If the resulting function \( g(\cdot) \) possesses a generalised harmonic analysis, then the recorded data can be regarded as coming from an asymptotically stationary time series. In the particular situation cited above, \( g(\cdot) \) is a periodic function (with period \( \alpha + \beta \)), and this is especially convenient since it is easy to see that periodicity is sufficient to ensure a generalised harmonic analysis. For if \( g(\cdot) \) has period \( P \),

\[
\frac{1}{T} \sum_{t=1}^{T-|\nu|} g(t) g(t+\nu) - \frac{1}{P} \sum_{t=1}^{P} g(t) g(t+|\nu|),
\]  

(3.1)

so that \( \text{Rg}(\nu) \) exists for all \( \nu \) and itself has period \( P \). In the case of periodically missing data, \( P = \alpha + \beta \) and
\[ R_g(v) = \frac{\alpha - \nu}{\alpha + \beta}, \quad \nu = 0, 1, \ldots, \beta \]
\[ = \frac{\alpha - \beta}{\alpha + \beta}, \quad \nu = \beta, \beta + 1, \ldots, \alpha \]
\[ = \frac{\nu - \beta}{\alpha + \beta}, \quad \nu = \alpha, \alpha + 1, \ldots, \alpha + \beta \quad (3.2) \]

assuming $\alpha > \beta$.

From (2.10), it follows that consistent estimates of the spectrum $f(\cdot)$ of $\{X_t\}$ are available as
\[ \hat{f}_T(\omega) = \frac{1}{2\pi T} \sum_{t=1}^{T} \sum_{s=1}^{T} k_T(t-s) h(t,s) X_t X_s \cos(t-s)\omega \quad (3.3) \]
where
\[ h(t,s) = \frac{g(t)g(s)}{R_g(t-s)} \quad (3.4) \]

It is shown in Parzen (1962) that an upper bound to the asymptotic variance of the estimate $\hat{f}_T(\omega)$ is given by
\[ \lim_{T \to \infty} \frac{1}{MT} \text{var}(\hat{f}_T(\omega)) \leq \max_{\omega} f(\omega)^2 \int_0^1 k(\theta)^2 d\theta \quad (3.5) \]
where in the case of a periodic amplitude modulating function,
\[ \bar{H} = \frac{1}{p^2} \sum_{t=1}^{T} \sum_{s=1}^{T} h(t,s)^2 \quad (3.6) \]
Equality in (3.5) is obtained when $\{X_t\}$ is a white noise process, i.e. $f(\cdot)$ is constant.

Due to the importance of the statistic $\bar{H}$, the next section is devoted to finding a simpler expression for it.

\textbf{§4. The Variance-Ratio, $\bar{H}$}

From (3.4) and (3.6),
\[ \overline{H} = \frac{1}{p^2} \sum_{t,s=1}^{p} \frac{(g(t)g(s))_2}{Rg(t-s)^2} \]

\[ = \frac{1}{p^2} \sum_{t,s=1}^{p} \frac{g(t)g(s)}{Rg(t-s)^2} \]  \hfill (4.1)

(since \( g(\cdot) \) takes only the values 0 and 1), where for \( 0 \leq \nu < p \),

\[ Rg(\nu) = \frac{1}{p} \sum_{t=1}^{p} g(t)g(t+\nu) \]

\[ = Rg(\nu+kp) , \quad k = 1, 2, 3, \ldots \]

It follows that \( \overline{H} \) may be written

\[ \overline{H} = \sum_{\nu=0}^{p-1} \frac{a_\nu}{Rg(\nu)^2} \]

for some set of constants \( \{a_\nu ; \quad \nu = 0, 1, \ldots, p-1\} \). We will now show that

\[ a_\nu = \frac{1}{p} Rg(\nu) \]

so that the formula for \( \overline{H} \) is considerably simplified.

**Theorem** \( \overline{H} = \frac{1}{p} \sum_{\nu=0}^{p-1} \frac{1}{Rg(\nu)} \).

**Proof:**

From (4.1),

\[ \overline{H} = \frac{1}{p^2} \sum_{t,s=1}^{p} \frac{g(t)g(s)}{Rg(t-s)^2} . \]

Writing \( \nu \) for \( t-s \),

\[ \overline{H} = \frac{1}{p^2} \sum_{t=1}^{p} \sum_{\nu=t-P}^{t-1} \frac{g(t)g(t-\nu)}{Rg(\nu)^2} \]

\[ = \frac{1}{p^2} \left( \sum_{\nu=0}^{p-1} \sum_{t=\nu+1}^{P} \frac{g(t)g(t-\nu)}{Rg(\nu)^2} + \sum_{\nu=-P+1}^{0} \sum_{t=1}^{\nu+1} \frac{g(t)g(t-\nu)}{Rg(\nu)^2} \right) \]
\[ = \frac{1}{p^2} \sum_{v=0}^{P-1} \frac{1}{Rg(v)^2} \sum_{t=v+1}^{P} g(t) g(t-v) \]
\[ + \frac{1}{p^2} \sum_{v=-P+1}^{-1} \frac{1}{Rg(v)^2} \sum_{t=1}^{P+v} g(t) g(t-v) \]
\[ = \frac{1}{p^2} \sum_{v=-P+1}^{P-1} b_v \cdot \frac{1}{Rg(v)^2} , \quad (4.2) \]

say. Now for \( v > 0 \),
\[ b_{-v} = \sum_{t=1}^{P-v} g(t) g(t+v) . \]

Writing \( u = t + v \), we have
\[ b_{-v} = \sum_{u=v+1}^{P} g(u-v) g(u) \]
\[ = b_v \]

so that \( b_v \) is an even function. Accordingly (4.2) can be written
\[ \bar{H} = \frac{b_0}{p^2 Rg(0)^2} + \frac{2}{p^2} \sum_{v=1}^{P-1} \frac{b_v}{Rg(v)^2} . \quad (4.3) \]

Further,
\[ b_{P-v} = \sum_{t=P-v+1}^{P} g(t) g(t-P+v) \]
\[ = \sum_{t=P-v+1}^{P} g(t-P) g(t-P+v) \]

since \( g(\cdot) \) has period \( P \), and writing \( u = t-P+v \), this is
\[ b_{P-v} = \sum_{u=1}^{v} g(u-v) g(u) . \quad (4.4) \]

It thus follows from (4.2) and (4.4) that
\[ b_v + b_{P-v} = \sum_{t=1}^{P} g(t) g(t-v) = PRg(v), \quad 0 < v < P \quad (4.5) \]
It may be immediately deduced from (4.5) that if \( P \) is even then

\[
b_{P/2} = \frac{P}{2} \text{Rg}\left(\frac{P}{2}\right)
\]  

(4.6)

and it should also be observed from (4.2) that

\[
b_0 = \sum_{t=1}^{P} g(t)^2 = P \text{Rg}(0).
\]  

(4.7)

Substituting (4.5), (4.6) and (4.7) into (4.3), and using the periodicity of \( \text{Rg}(\cdot) \), we obtain the final result that

\[
\overline{H} = \frac{1}{P} \sum_{\nu=0}^{P-1} \frac{1}{\text{Rg}(\nu)}.
\]  

(4.8)

It is easy to see from (4.8) that \( \overline{H} \) is a reasonable measure of the variance-ratio in that if no data is missing, \( \text{Rg}(\cdot) \equiv 1 \) and hence

\( \overline{H} = 1 \), and when there is missing data, \( \text{Rg}(\cdot) < 1 \) and \( \overline{H} > 1 \).

§5. Extension of the estimation frequency range by using two data recorders

We now use the material of the preceding paragraphs to obtain and study a method of estimating the spectrum over a period range from \( \infty \) down to a smaller value than the natural one which is twice the sampling period (see §1). It is assumed that there are available two like mechanisms for recording the time series, each of which has a minimum possible sampling period of \( \Delta t \) and can be set to read the data at any sampling period greater than or equal to \( \Delta t \). The motivation behind this is that a mechanism requires a certain finite amount of time (\( \Delta t \)) to read and record a value; thus it is impractical to read data more frequently than with period \( \Delta t \); however no great difficulty should be caused by requiring that an additional pause-time be included over and above this natural lower limit. It is further assumed that the value recorded is an accurate reading at the specified time, and not, for example, some weighted average over an interval of time which is of the same order as \( \Delta t \).
\[ R_g(i \rho_2) = \frac{1}{\rho_2}, \quad i \neq j \rho_1 \]

\[ R_g(v) = \frac{2}{\rho_1 \rho_2}, \quad \text{otherwise.} \quad (5.3) \]

It is apparent that for most choices of \( \rho_1 \) and \( \rho_2 \), a considerable majority of the data is missing, so that to obtain a satisfactory estimate of the spectrum, the sample must be rather large. Considering the statistic \( \overline{H} \) (the asymptotic maximum variance ratio), from (4.8),

\[ \overline{H} = \frac{1}{\rho_1 \rho_2} \sum_{v=0}^{\rho_1 \rho_2 - 1} \frac{1}{R_g(v)} \]

\[ = \frac{1}{\rho_1 \rho_2} \left[ \frac{\rho_1 \rho_2}{\rho_1 + \rho_2 - 1} + (\rho_1 - 1)\rho_2 + (\rho_2 - 1)\rho_1 + \frac{1}{2} \rho_1 \rho_2 (\rho_1 \rho_2 - \rho_1 - \rho_2 + 1) \right] \]
= \frac{1}{\rho_1^* + \rho_2} + 1 - \frac{1}{\rho_1} + 1 - \frac{1}{\rho_2} + \frac{1}{2}(\rho_1^* \rho_2 - \rho_1 - \rho_2^* + 1) \\
= \frac{1}{\rho_1^* + \rho_2} - \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) + \frac{1}{2}(\rho_1^* \rho_2 - \rho_1 - \rho_2^*) + \frac{5}{2}. \quad (5.4)

Neglecting the first two terms (whose contribution is not greater than \(\frac{7}{12}\) in magnitude), \(\bar{H}\) may be approximated by

\[\frac{1}{2}(\rho_1^* \rho_2 - \rho_1 - \rho_2^* + 5). \quad (5.5)\]

This approximation gives the integer value immediately above the true value of \(\bar{H}\), and the error of the approximation is less than \(\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\) in magnitude.

The Table gives the values of \(\bar{H}\) for all relatively prime pairs of integers not exceeding 15. Also included are the values of \(\bar{H}^{1/2}\), which is a measure of the maximum proportional increased value of the standard deviation, the proportion of the observable sample which is read, i.e.

\[R_{g(0)} = \frac{\rho_1 + \rho_2 - 1}{\rho_1 \rho_2} \quad (5.6)\]

and \(\bar{H}^{1/2} / R_{g(0)}\). As can be seen from the values of this last statistic, unless \(\rho_1\) is much less than \(\rho_2\), the increased value of the standard deviation is

\[S_\theta / R_{g(0)} \quad (5.7)\]

where \(S\) is the value of the standard deviation when no data is missing, and \(\theta\) is generally between 1.2 and 1.3 in the range tabulated. From (5.5) and (5.6), it can be seen that if

\[\rho_1 + \gamma \rho_2 \quad , \quad 0 < \gamma < 1 \quad (5.8)\]
as \( \rho_1, \rho_2 \rightarrow \infty \), then

\[
Rg(0) \frac{H^{1/2}}{(2\gamma)^{1/2}} + \frac{1+\gamma}{(2\gamma)^{1/2}}. \tag{5.9}
\]

In practice the use of \( \theta = \frac{1+\gamma}{(2\gamma)^{1/2}} \) in (5.7) over-estimates the value of the standard deviation.

As has been illustrated, the cases \( \rho_1 = \rho_2 - 1 \) are of special importance. For these cases, \( \gamma \rightarrow 1 \), and so

\[
Rg(0) \frac{H^{1/2}}{\sqrt{2}} \tag{5.10}
\]

and since

\[
Rg(0) = \frac{2}{\rho_2} \tag{5.11}
\]

here, it follows that the factor by which the standard deviation is multiplied is not greater than \( 2^{-1/2} \rho_2 \).

The equivalent sample size for these estimates is \( \frac{H^{-1}}{T} \) times the observable sample size \( T \). For the effect of this form of missing data is to multiply the variance by up to \( H \), and this is also the effect of reducing the size of the classical form of sample by a factor of \( H \), since the variance is inversely proportional to the sample size (see (1.11)) if the truncation point \( M_1 \) is held constant. The fact that the equivalent sample size is so small is not surprising when it is remembered that for every segment of length \( \rho_1 \rho_2 \) of the "observable sample", there are at most two terms which contribute to the estimate of \( R(\nu) \) if \( \nu \) is not a multiple of \( \rho_1 \) or \( \rho_2 \), and such \( \nu \)'s are invariably in the majority.
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REFERENCES


§6. Other missing data problems.

The work presented in this paper, apart from §5, applies to a larger class of missing data situations, viz. those with periodic amplitude modulating functions. These and other missing data problems are discussed in detail in Neave (1966).