# Multilevel Models

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# Reminder about Corn Example

- We consider a subset of a larger data set on corn grown on the island Antigua.
- The response variable we consider is the harvest weight (`harvwt`) per plot (units unknown).
- There are eight sites with eight separate plots within each site where the corn is grown under the same treatment conditions.

# Multilevel Model

- In a multilevel model, we may have

$$y_i = \alpha_{j[i]} + e_i$$

where $i = 1, \ldots, 64$ indexes the observation and $j[i] = 1, \ldots, 8$ indicates which of the eight sites contains the $i$th observation.

  - $\alpha_j, \sim N(\mu_\alpha, \sigma_\alpha^2)$ are *modeled*;
  - $e_i \sim \text{iid } N(0, \sigma^2)$;
  - $\mu_\alpha$, $\sigma_\alpha$, and $\sigma^2$ are unmodeled.

# Likelihood

- We can express the likelihood of the data in terms of the unmodeled and modeled parameters.
- The normal density $f(x \mid \mu, \sigma)$ for mean $\mu$ and standard deviation $\sigma$ is large when the $x$ is close to $\mu$.
- The likelihood has the form

$$L = \left( \prod_{j=1}^{J} f(\alpha_j \mid \mu_\alpha, \sigma_\alpha) \right) \left( \prod_{i=1}^{n} f(y_i \mid \alpha_{j[i]}, \sigma) \right)$$

- The left factor is large when all of the $\alpha_j$ are close to $\mu_\alpha$.
- The right factor is large when each $\alpha_j$ is close to the $y_i$ in group $j$.
- The best estimate of $\mu_\alpha$ will be the overall mean $\bar{y}_{\text{all}}$.
- The likelihood is maximized at a balancing point when the $\alpha_j$ is somewhere between the overall mean $\bar{y}_{\text{all}}$ and the sample means $\bar{y}_j$.

# Pooling

- *Complete pooling* is the estimate that assumes no difference between groups so that
$$\alpha_1 = \alpha_2 = \cdots = \alpha_J = \bar{y}_{\text{all}}$$
This corresponds to an extreme multilevel model where $\sigma_\alpha = 0$.
- *No pooling* estimates each $\alpha_j$ using only data from the $j$th sample.
$$\alpha_1 = \bar{y}_j, \ \ldots, \ \alpha_J = \bar{y}_J$$
This corresponds to an extreme multilevel model where $\sigma_\alpha = +\infty$.
- Multilevel models correspond to *partial pooling* where data from other samples effects estimates for sample $j$, but the data within sample $j$ is most influential.
- The estimated coefficients are a weighted average between the corresponding sample mean and the grad mean.
$$\hat{\alpha}_j \approx \frac{\left(\frac{n_j}{\sigma^2}\right)\bar{y}_j + \left(\frac{1}{\sigma_\alpha^2}\right)\bar{y}_{\text{all}}}{\frac{n_j}{\sigma^2} + \frac{1}{\sigma_\alpha^2}}$$

# Weighted Average

$$\hat{\alpha}_j \approx \frac{\left(\frac{n_j}{\sigma^2}\right)\bar{y}_j + \left(\frac{1}{\sigma_\alpha^2}\right)\bar{y}_{\text{all}}}{\frac{n_j}{\sigma^2} + \frac{1}{\sigma_\alpha^2}} = \left(\frac{\frac{n_j}{\sigma^2}}{\frac{n_j}{\sigma^2} + \frac{1}{\sigma_\alpha^2}}\right)\bar{y}_j + \left(\frac{\frac{1}{\sigma_\alpha^2}}{\frac{n_j}{\sigma^2} + \frac{1}{\sigma_\alpha^2}}\right)\bar{y}_{\text{all}}$$

- An expression of the form

$$w_1 A + w_2 B$$

  where $w_1 + w_2 = 1$ is a *weighted average* of $A$ and $B$.
- The relative distance of the average to the endpoints is inversely proportional to the weights.
- (If one weight is ten times the other, the distance of the average to the end with the higher weight will be ten times smaller than to the other.)

# Interpretations

$$\hat{\alpha}_j \approx \frac{\left(\frac{n_j}{\sigma^2}\right)\bar{y}_j + \left(\frac{1}{\sigma_\alpha^2}\right)\bar{y}_{\text{all}}}{\frac{n_j}{\sigma^2} + \frac{1}{\sigma_\alpha^2}}$$

- Here, when:
  - $n_j$ gets bigger (more direct data in the group), or;
  - $\sigma^2$ gets smaller (more likely for $y_i$ to be close to group means)

  $\hat{\alpha}_j$ moves closer to $\bar{y}_j$.
- When
  - $\sigma_\alpha^2$ gets smaller (more likely for all $\alpha_j$ to be close to each other),

  $\hat{\alpha}_j$ moves closer to $\bar{y}_{\text{all}}$.

# Numerical Example

- $n_j = 8$ for $j = 1, \ldots, 8$
- $\hat{\mu}_\alpha = \bar{y}_{\text{all}} = 4.29$
- $\hat{\sigma}_\alpha = 1.55$
- $\hat{\sigma} = 0.87$
- $\bar{y}_1 = 4.88$
- $\hat{\alpha}_1 = 4.86$

# Site DBAN

- $\frac{8}{(0.87)^2} = 10.57$
- $\frac{1}{(1.55)^2} = 0.42$
- $$\frac{10.57}{10.99}(4.88) + \frac{0.42}{10.99}(4.29) = 4.86$$
- The estimate is shrunk very little toward the grand mean.

# Site NSAN

- $\bar{y}_3 = 2.09$
- $\bar{y}_{\text{all}} = 4.29$
- $$\frac{10.57}{10.99}(2.09) + \frac{0.42}{10.99}(4.29) = 2.17$$
- The estimate is shrunk a little more toward the grand mean than for the other site since the same proportion of a longer distance is larger.