

Simple Linear Regression

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Phosphorous Example

- Researchers gathered data to evaluate the use of phosphorus (P) by nine corn plants.
- The data consist of x , the inorganic P in soil (ppm), and y , the plant-available P (ppm).

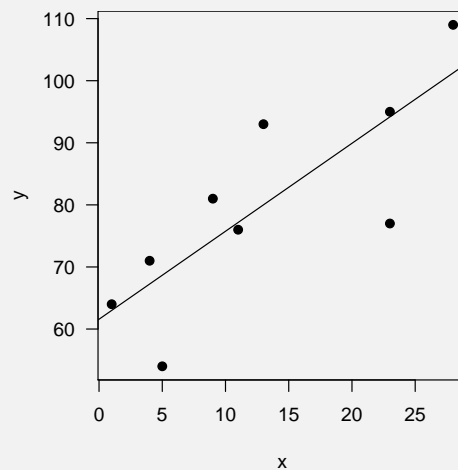
x	1	4	5	9	13	11	23	23	28
y	64	71	54	81	93	76	77	95	109

Exploratory data analysis

- Graphical summaries (scatter plot)
- Numerical summaries (means, sds, correlation)
- Use R!

Graphical exploration of two quantitative variables

```
> x = c(1, 4, 5, 9, 13, 11, 23, 23, 28)
> y = c(64, 71, 54, 81, 93, 76, 77, 95, 109)
> plot(x, y, pch = 16, las = 1)
> fit = lm(y ~ x)
> abline(fit)
```



Objectives of simple linear regression

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- Description** To describe the relationship between inorganic P in soil and plant-available P
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- Estimation** To estimate the population mean plant-available P level at a given level of inorganic P in soil
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- Prediction** To predict the plant-available P level for an individual plant at a given level of inorganic P in soil
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- Testing** To test if there is a relationship between inorganic P in soil and plant-available P
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Simple Linear Regression Model

- $y_i = \beta_0 + \beta_1 x_i + e_i$, $e_i \sim \text{iid } N(0, \sigma^2), i = 1, \dots, n$
- $y = \beta_0 + \beta_1 x$ is the “true regression line”
- β_0 is the intercept, β_1 is the slope
- x_i is the explanatory variable
- y_i is the response variable
- e_i is random error
- iid stands for *independent and identically distributed*

Simple Linear Regression Assumptions

- 1 The model is correct: $E(y_i) = \beta_0 + \beta_1 x_i$.
- 2 Errors e_i are independent.
- 3 Errors e_i have homogeneous variance: $\text{Var}(e_i) = \sigma^2$.
- 4 Errors e_i have normal distribution: $e_i \sim N(0, \sigma^2)$.

Estimating Model Parameters

- A well estimated line should be “close to the data points”.
- The *least squares criterion* says that best line is the one that minimizes $\sum_{i=1}^n (y_i - (b_0 + b_1 x_i))^2$.
- The solution to this problem is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- The fitted values are $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$
- The estimated variance is $\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2$

An Alternative Viewpoint

- The *correlation coefficient* r is a number between -1 and 1 that measures the *strength of the linear relationship* between x and y .
- $$r = \frac{1}{n-1} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right)$$
- The estimated y for an x that is z standard deviations from the mean is rz standard deviations from the mean.
- In other words, $\hat{y} = \bar{y} + rz s_y$.
- The estimated slope and intercept are:

$$\hat{\beta}_1 = r \frac{s_y}{s_x}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
- The regression line goes through the point (\bar{x}, \bar{y}) .

Hypothesis Testing: $H_0: \beta_1 = 0$

- The test statistic $t = \frac{\hat{\beta}_1 - 0}{\text{SE}(\hat{\beta}_1)} \sim T_{n-2}$ under $H_0: \beta_1 = 0$.
- The p-value is the area under the t -distribution more extreme than the observed t .

ANOVA Approach

Sum of squares	Expression	Degrees of freedom
<i>Total (SSTot)</i>	$\sum_{i=1}^n (y_i - \bar{y})^2$	$n - 1$
<i>Regression (SSReg)</i>	$\sum_{i=1}^n (\hat{y}_i - \bar{y})^2$	1
<i>Error (SSErr)</i>	$\sum_{i=1}^n (y_i - \hat{y}_i)^2$	$n - 2$

Partition: SSTot = SSReg + SSErr.

ANOVA Table

Source	df	SS	MS	F
Regression	1	SSReg	SSReg/df	MSReg/MSErr
Error	n-2	SSErr	SSErr/df	
Total	n-1	SSTot		

$$F = \frac{\text{MSReg}}{\text{MSErr}} \sim F_{1, n-2} \text{ under } H_0: \beta_1 = 0$$

Simple Linear Regression in R

```
> soilP = c(1, 4, 5, 9, 13, 11, 23, 23, 28)
> plantP = c(64, 71, 54, 81, 93, 76, 77, 95, 109)
> plot(soilP, plantP)
> fit = lm(plantP ~ soilP)
> coef(fit)
> summary(fit)
> anova(fit)
> plot(fitted(fit), residuals(fit))
```