

Nonparametric Methods for Two Samples

An overview

- In the independent two-sample t-test, we assume normality, independence, and equal variances.
- This t-test is robust against nonnormality, but is sensitive to dependence.
- If n_1 is close to n_2 , then the test is moderately robust against unequal variance ($\sigma_1^2 \neq \sigma_2^2$). But if n_1 and n_2 are quite different (e.g. differ by a ratio of 3 or more), then the test is much less robust.
- How to determine whether the equal variance assumption is appropriate?
- Under normality, we can compare σ_1^2 and σ_2^2 using S_1^2 and S_2^2 , but such tests are very sensitive to nonnormality. Thus we avoid using them.
- Instead we consider a *nonparametric test* called Levene's test for comparing two variances, which does not assume normality while still assuming independence.
- Later on we will also consider nonparametric tests for comparing two means.

193

Nonparametric Methods for Two Samples

Levene's test

Consider two independent samples Y_1 and Y_2 :

Sample 1: 4, 8, 10, 23

Sample 2: 1, 2, 4, 4, 7

Test $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_A : \sigma_1^2 \neq \sigma_2^2$.

- Note that $s_1^2 = 67.58$, $s_2^2 = 5.30$.
- The main idea of Levene's test is to turn testing for equal variances using the original data into testing for equal means using modified data.
- Suppose normality and independence, if Levene's test gives a small p-value (< 0.01), then we use an approximate test for $H_0 : \mu_1 = \mu_2$ vs $H_A : \mu_1 \neq \mu_2$. See Section 10.3.2 of the bluebook.

194

Nonparametric Methods for Two Samples

Levene's test

- (1) Find the median for each sample. Here $\tilde{y}_1 = 9$, $\tilde{y}_2 = 4$.
- (2) Subtract the median from each obs.

Sample 1: -5, -1, 1, 14
Sample 2: -3, -2, 0, 0, 3

- (3) Take absolute values of the results.

Sample 1*: 5, 1, 1, 14
Sample 2*: 3, 2, 0, 0, 3

- (4) For any sample that has an odd sample size, remove 1 zero.

Sample 1*: 5, 1, 1, 14
Sample 2*: 3, 2, 0, 3

- (5) Perform an independent two-sample t-test on the modified samples, denoted as Y_1^* and Y_2^* . Here $\bar{y}_1^* = 5.25$, $\bar{y}_2^* = 2$, $s_1^{2*} = 37.58$, $s_2^{2*} = 2.00$. Thus $s_p^2 = 19.79$, $s_p = 4.45$ on df = 6 and the observed

$$t = \frac{5.25 - 2}{4.45\sqrt{1/4 + 1/4}} = 1.03$$

on df = 6. The p-value $2 \times P(T_6 \geq 1.03)$ is more than 0.20. Do not reject H_0 at the 5% level.

195

Nonparametric Methods for Two Samples

Mann-Whitney test

- We consider a nonparametric Mann-Whitney test (aka Wilcoxon test) for independent two samples, although analogous tests are possible for paired two samples.
- We relax the distribution assumption, but continue to assume independence.
- The main idea is to base the test on the ranks of obs.
- Consider two independent samples Y_1 and Y_2 :

Sample 1: 11, 22, 14, 21

Sample 2: 20, 9, 12, 10

Test $H_0 : \mu_1 = \mu_2$ vs $H_A : \mu_1 \neq \mu_2$.

196

Nonparametric Methods for Two Samples

Mann-Whitney test

(1) Rank the obs

rank	obs	sample
1	9	2
2	10	2
3	11	1
4	12	2
5	14	1
6	20	2
7	21	1
8	22	1

(2) Compute the sum of ranks for each sample. Here $RS(1) = 3 + 5 + 7 + 8 = 23$ and $RS(2) = 1 + 2 + 4 + 6 = 13$.

(3) Under H_0 , the means are equal and thus the rank sums should be about equal. To compute a p-value, we list all possible ordering of 8 obs and find the rank sum of each possibility. Then p-value is $2 \times P(RS(2) \leq 13)$. Here

$$\begin{aligned} P(RS(2) \leq 13) &= P(RS(2) = 10) + P(RS(2) = 11) \\ &\quad + P(RS(2) = 12) + P(RS(2) = 13) \\ &= 7/70 = 0.1 \end{aligned}$$

and thus p-value = 0.2.

Nonparametric Methods for Two Samples

Mann-Whitney test

- If we had observed 10, then p-value = $2 \times 1/70 = 0.0286$.
- If we had observed 11, then p-value = $2 \times 2/70 = 0.0571$.
- Thus for this sample size, we can only reject at 5% if the observed rank sum is 10.
- Table A10 gives the cut-off values for different sample sizes. For $n_1 = n_2 = 4$ and $\alpha = 0.05$, we can only reject H_0 if the observed rank sum is 10.

Nonparametric Methods for Two Samples

Mann-Whitney test

Recorded below are the longevity of two breeds of dogs.

Breed	A	Breed	B
obs	rank	obs	rank
12.4	9	11.6	7
15.9	14	9.7	4
11.7	8	8.8	3
14.3	11.5	14.3	11.5
10.6	6	9.8	5
8.1	2	7.7	1
13.2	10		
16.6	15		
19.3	16		
15.1	13		
$n_2 = 10$		$n_1 = 6$	
		$T^* = 31.5$	

Nonparametric Methods for Two Samples

Mann-Whitney test

- Here n_1 is the sample size in the smaller group and n_2 is the sample size in the larger group.
- T^* is the sum of ranks in the smaller group. Let $T^{**} = n_1(n_1 + n_2 + 1) - T^* = 6 \times 17 - 31.5 = 70.5$.
- Let $T = \min(T^*, T^{**}) = 31.5$ and look up Table A10.
- Since the observed T is between 27 and 32, the p-value is between 0.01 and 0.05. Reject H_0 at 5%.

Remarks

- If there are ties, Table A10 gives approximation only.
- The test does not work well if the variances are very different.
- It is not easy to extend the idea to more complex types of data. There is no CI.
- For paired two samples, consider using signed rank test.
- See p.251 of the bluebook for a decision tree.

Nonparametric Methods for Two Samples

Key R commands

```
> # Levene's test
> levene.test = function(data1, data2){
+ levene.trans = function(data){
+ a = sort(abs(data-median(data)))
+ if (length(a)%2)
+ a[1]=0[duplicated(a)]
+ else a
+ }
+ t.test(levene.trans(data1), levene.trans(data2), var.equal=T)
+ }
> y1 = c(4,8,10,23)
> y2 = c(1,2,4,4,7)
> levene.test(y1, y2)
```

Two Sample t-test

```
data: levene.trans(data1) and levene.trans(data2)
t = 1.0331, df = 6, p-value = 0.3414
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
-4.447408 10.947408
sample estimates:
mean of x mean of y
5.25      2.00
```

```
>
> # Mann-Whitney test example
> samp1 = c(11, 22, 14, 21)
> samp2 = c(20, 9, 12, 10)
> # W = 23-10 = 13
> wilcox.test(samp1, samp2)
```

Wilcoxon rank sum test

```
data: samp1 and samp2
W = 13, p-value = 0.2
alternative hypothesis: true mu is not equal to 0
```

201

```
>
> breedA = c(12.4, 15.9, 11.7, 14.3, 10.6, 8.1, 13.2, 16.6, 19.3, 15.1)
> breedB = c(11.6, 9.7, 8.8, 14.3, 9.8, 7.7)
> # W = 70.5-21 = 49.5
> wilcox.test(breedA, breedB)

Wilcoxon rank sum test with continuity correction

data: breedA and breedB
W = 49.5, p-value = 0.03917
alternative hypothesis: true mu is not equal to 0

Warning message:
Cannot compute exact p-value with ties in: wilcox.test.default(breedA, breedB)
>
```

202

Comparing Two Proportions

Test procedure

Consider two binomial distributions $Y_1 \sim B(n_1, p_1)$, $Y_2 \sim B(n_2, p_2)$, and Y_1, Y_2 are independent. We want to test

$$H_0 : p_1 = p_2 \quad \text{vs} \quad H_A : p_1 \neq p_2$$

- Use the point estimator $\hat{p}_1 - \hat{p}_2$, where $\hat{p}_1 = Y_1/n_1$, $\hat{p}_2 = Y_2/n_2$ are the sample proportions.
- Note that $\mu_{\hat{p}_1 - \hat{p}_2} = E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2$ and $\sigma_{\hat{p}_1 - \hat{p}_2}^2 = \text{Var}(\hat{p}_1 - \hat{p}_2) = p_1(1 - p_1)/n_1 + p_2(1 - p_2)/n_2$.
- Under $H_0 : p_1 = p_2 = p$, $\mu_{\hat{p}_1 - \hat{p}_2} = 0$ and $\sigma_{\hat{p}_1 - \hat{p}_2}^2 = p(1 - p)(1/n_1 + 1/n_2)$.
- Under H_0 , the test statistic is approximately normal,

$$Z = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{p(1 - p)(1/n_1 + 1/n_2)}} \approx N(0, 1)$$

- But we do not know p and thus estimate it by

$$\hat{p} = \frac{Y_1 + Y_2}{n_1 + n_2}$$

- Thus the test statistic is $Z = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\hat{p}(1 - \hat{p})(1/n_1 + 1/n_2)}} \approx N(0, 1)$ under H_0 .

203

Comparing Two Proportions

Potato cure rate example

A plant pathologist is interested in comparing the effectiveness of two fungicide used on infested potato plants. Let Y_1 denote the number of plants cured using fungicide A among n_1 plants and let Y_2 denote the number of plants cured using fungicide B among n_2 plants. Assume that $Y_1 \sim B(n_1, p_1)$ and $Y_2 \sim B(n_2, p_2)$, where p_1 is the cure rate of fungicide A and p_2 is the cure rate of fungicide B. Suppose the obs are $n_1 = 105, p_1 = 71$ for fungicide A and $n_2 = 87, p_2 = 45$ for fungicide B. Test $H_0 : p_1 = p_2$ vs $H_A : p_1 \neq p_2$.

- Here $\hat{p}_1 = 71/105 = 0.676$, $\hat{p}_2 = 45/87 = 0.517$, and the pooled estimate of cure rate is

$$\hat{p} = \frac{71 + 45}{105 + 87} = 0.604$$

- Thus the observed test statistic is

$$z = \frac{(0.676 - 0.517) - 0}{\sqrt{0.604 \times 0.396 \times (1/105 + 1/87)}} = 2.24$$

- Compared to Z , the p-value is $2 \times P(Z \geq 2.24) = 0.025$.
- Reject H_0 at the 5% level. There is moderate evidence against H_0 .

204

Comparing Two Proportions

Remarks

- For constructing a $(1 - \alpha)$ CI for $p_1 - p_2$, there is no H_0 . Since $Var(\hat{p}_1 - \hat{p}_2) = p_1(1 - p_1)/n_1 + p_2(1 - p_2)/n_2$, estimate by

$$\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}$$

and the CI is

$$\hat{p}_1 - \hat{p}_2 - z_{\alpha/2} \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}} \leq p_1 - p_2 \leq \hat{p}_1 - \hat{p}_2 + z_{\alpha/2} \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

- In the potato cure rate example, a 95% CI for $p_1 - p_2$ is

$$(0.676 - 0.517) \pm 1.96 \times \sqrt{\frac{0.676 \times 0.324}{105} + \frac{0.517 \times 0.483}{87}}$$

which is 0.159 ± 0.138 or $[0.021, 0.297]$.

- In constructing CI for $p_1 - p_2$, normal approximation works well if $n_1\hat{p}_1 \geq 5, n_1(1 - \hat{p}_1) \geq 5, n_2\hat{p}_2 \geq 5, n_2(1 - \hat{p}_2) \geq 5$.
- In testing $H_0 : p_1 = p_2$, normal approximation works well if $n_1\hat{p} \geq 5, n_1(1 - \hat{p}) \geq 5, n_2\hat{p} \geq 5, n_2(1 - \hat{p}) \geq 5$.

205

Comparing Two Proportions

Key R commands

```
> # potato cure rate example
> y1 = 71
> n1 = 105
> y2 = 45
> n2 = 87
> p1 = y1/n1
> p2 = y2/n2
> poolp = (y1+y2)/(n1+n2)
> poolp
[1] 0.6041667
> z.value = (p1-p2)/sqrt(poolp*(1-poolp)*(1/n1+1/n2))
> z.value
[1] 2.241956
> # p-value
> 2*pnorm(z.value, lower.tail=F)
[1] 0.02496419
> # 95% CI
> alpha = 0.05
> qnorm(alpha/2, lower.tail=F)*sqrt(p1*(1-p1)/n1+p2*(1-p2)/n2)
[1] 0.1379716
> c(p1-p2-qnorm(alpha/2, lower.tail=F)*sqrt(p1*(1-p1)/n1+p2*(1-p2)/n2),
+ p1-p2+qnorm(alpha/2, lower.tail=F)*sqrt(p1*(1-p1)/n1+p2*(1-p2)/n2))
[1] 0.02097751 0.29692068
>
> prop.test(c(71, 45), c(105, 87), correct=F)

      2-sample test for equality of proportions without continuity
      correction

data:  c(71, 45) out of c(105, 87)
X-squared = 5.0264, df = 1, p-value = 0.02496
alternative hypothesis: two.sided
95 percent confidence interval:
 0.02097751 0.29692068
sample estimates:
 prop 1 prop 2
0.6761905 0.5172414
```

206

```
>
> prop.test(c(71, 45), c(105, 87))

      2-sample test for equality of proportions with continuity correction

data:  c(71, 45) out of c(105, 87)
X-squared = 4.3837, df = 1, p-value = 0.03628
alternative hypothesis: two.sided
95 percent confidence interval:
 0.01046848 0.30742971
sample estimates:
 prop 1 prop 2
0.6761905 0.5172414
```

207

One-way ANOVA

An overview

- So far we have learned statistical methods for comparing two trts.
- One-way analysis of variance (ANOVA) provides us with a way to compare more than two trts.
- One-way ANOVA can be viewed as an extension of the independent two sample case to independent multiple samples.
- The key idea is to break up the sum of squares

$$\sum (Y_i - \bar{Y})^2$$

- First reconsider the independent two-sample case and then generalize the idea to independent multiple samples.

208

One-way ANOVA

Independent two samples

- Consider the following independent two samples:

X: 4, 12, 8

Y: 17, 8, 11

- The summary statistics are

$$\bar{x} = 8, \quad s_x^2 = 16, \quad \sum_{i=1}^3 (x_i - \bar{x})^2 = 32$$

$$\bar{y} = 12, \quad s_y^2 = 21, \quad \sum_{i=1}^3 (y_i - \bar{y})^2 = 42, \quad s_p^2 = 18.5$$

- For testing $H_0 : \mu_1 = \mu_2$ vs $H_A : \mu_1 \neq \mu_2$, use t-test

$$t = \frac{(12 - 8) - 0}{\sqrt{18.5(1/3 + 1/3)}} = 1.14$$

on $df = 4$. The p-value $2 \times P(T_4 \geq 1.14)$ is great than 0.10. Thus do not reject H_0 at 5% and there is no evidence against H_0 .

- Now we will examine this using the idea of breaking up sums of squares.

209

One-way ANOVA

Sums of squares (SS)

- Total SS: Pretend that all obs are from a single population. The overall mean is

$$\frac{4 + 12 + 8 + 17 + 8 + 11}{6} = 10$$

and the SS Total is

$$(4-10)^2 + (12-10)^2 + (8-10)^2 + (17-10)^2 + (8-10)^2 + (11-10)^2 = 98$$

on $df = 5$.

- Treatment SS: How much of the total SS can be attributed to the differences between the two trt groups? Replace each obs by its group mean.

X: 8, 8, 8

Y: 12, 12, 12

The overall mean here is

$$\frac{8 + 8 + 8 + 12 + 12 + 12}{6} = 10$$

and the SS Trt is

$$(8-10)^2 + (8-10)^2 + (8-10)^2 + (12-10)^2 + (12-10)^2 + (12-10)^2 = 24$$

on $df = 1$.

210

One-way ANOVA

Sums of squares (SS)

- Error SS: How much of the total SS can be attributed to the differences within each trt group? The SS Error is

$$(4-8)^2 + (12-8)^2 + (8-8)^2 + (17-12)^2 + (8-12)^2 + (11-12)^2 = 74$$

on $df = 4$.

- Note that $SSE_{\text{Error}}/df = 74/4 = 18.5 = s_p^2$.

- Note also that

$$\begin{aligned} \text{SS Total} &= \text{SS Trt} + \text{SS Error} \quad (98 = 24 + 74) \\ \text{df Total} &= \text{df Trt} + \text{df Error} \quad (5 = 1 + 4) \end{aligned}$$

- An ANOVA table summarizes the information.

Source	df	SS	MS
Trt	1	24	24
Error	4	74	18.5
Total	5	98	-

- Here $MS = SS/df$.

211

One-way ANOVA

F-test

- $H_0 : \mu_1 = \mu_2$ vs $H_A : \mu_1 \neq \mu_2$

- A useful fact is that, under H_0 , the test statistic is:

$$F = \frac{MST_{\text{Trt}}}{MSE_{\text{Error}}} \sim F_{df_{\text{Trt}}, df_{\text{Error}}}$$

- In the example, the observed $f = 24/18.5 = 1.30$.
- Compare this to an F-distribution with 1 df in the numerator and 4 df in the denominator using Table D. The (one-sided) p-value $P(F_{1,4} \geq 1.30)$ is greater than 0.10. Do not reject H_0 at the 10% level. There is no evidence against H_0 .
- Note that a small difference between the two trt means relative to variability is associated with a small f , a large p-value, and accepting H_0 , whereas a large difference between the two trt means relative to variability is associated with a large f , a small p-value, and rejecting H_0 .
- Note that $f = 1.30 = (1.14)^2 = t^2$. That is $f = t^2$, but only when the df in the numerator is 1.
- Note that the p-value is one-tailed, even though H_A is two-sided.

212

One-way ANOVA

A recap

In the simple example above, there are 2 trts and 3 obs/trt. The overall mean is 10,

$$SSTotal = \sum_{i=1}^3 (x_i - 10)^2 + \sum_{i=1}^3 (y_i - 10)^2 = 98$$

$$SSTrt = 3 \times (\bar{x} - 10)^2 + 3 \times (\bar{y} - 10)^2 = 24$$

$$SSError = \sum_{i=1}^3 (x_i - 8)^2 + \sum_{i=1}^3 (y_i - 12)^2 = 74$$

with df = 5, 1, and 4, respectively.

213

One-way ANOVA

Generalization to k independent samples

- Consider k trts and n_i obs for the i^{th} trt.
- Let y_{ij} denote the j^{th} obs in the i^{th} trt group.
- Tabulate the obs as follows.

Trt	1	2	...	k		Trt	1	2	3	
Obs	y_{11}	y_{21}	...	y_{k1}			10	9	6	
	y_{12}	y_{22}	...	y_{k2}			7	12	2	
	\vdots	\vdots		\vdots			8	6	4	
	y_{1n_1}	y_{2n_2}	...	y_{kn_k}			12	7		
Sum	$y_{1.}$	$y_{2.}$...	$y_{k.}$	$y_{..}$	Sum	37	27	28	92
Mean	$\bar{y}_{1.}$	$\bar{y}_{2.}$...	$\bar{y}_{k.}$	$\bar{y}_{..}$	Mean	9.25	9	5.6	7.67

- Sum for the i^{th} trt: $y_{i.} = \sum_{j=1}^{n_i} y_{ij}$
- Mean for the i^{th} trt: $\bar{y}_{i.} = y_{i.}/n_i$
- Grand sum: $y_{..} = \sum_{i=1}^k \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^k y_{i.}$
- Grand mean: $\bar{y}_{..} = y_{..}/N$ where the total # of obs is:

$$N = \sum_{i=1}^k n_i = n_1 + n_2 + \dots + n_k.$$

214

One-way ANOVA

Basic partition of SS

$$\begin{aligned} SS \text{ Total} &= SS \text{ Trt} + SS \text{ Error} \\ df \text{ Total} &= df \text{ Trt} + df \text{ Error} \end{aligned}$$

where

$$SS \text{ Total} = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^k \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$df \text{ Total} = N - 1$$

$$SS \text{ Trt} = \sum_{i=1}^k n_i (\bar{y}_{i.} - \bar{y}_{..})^2 = \sum_{i=1}^k \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N}$$

=

$$df \text{ Trt} = k - 1$$

$$SS \text{ Error} = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2$$

$$= (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \dots + (n_k - 1)s_k^2$$

$$df \text{ Error} = N - k = (n_1 - 1) + \dots + (n_k - 1)$$

or simply $SS \text{ Error} = SS \text{ Total} - SS \text{ Trt}$ and $df \text{ Error} = df \text{ Total} - df \text{ Trt}$.

215

One-way ANOVA

Fish length example

- Consider the length of fish (in inch) that are subject to one of three types of diet, with seven observations per diet group. The raw data are:

Y_1	18.2	20.1	17.6	16.8	18.8	19.7	19.1
Y_2	17.4	18.7	19.1	16.4	15.9	18.4	17.7
Y_3	15.2	18.8	17.7	16.5	15.9	17.1	16.7

- A stem and leaf display of these data looks like:

	Y_1	Y_2	Y_3
15.		9	29
16.	8	4	57
17.	6	47	71
18.	28	74	8
19.	71	1	
20.	1		

- Summary statistics are:

$$\begin{aligned} y_1. &= 130.3 & \bar{y}_{1.} &= 18.61 & s_1^2 &= 1.358 & n_1 &= 7 \\ y_2. &= 123.6 & \bar{y}_{2.} &= 17.66 & s_2^2 &= 1.410 & n_2 &= 7 \\ y_3. &= 117.9 & \bar{y}_{3.} &= 16.84 & s_3^2 &= 1.393 & n_3 &= 7 \\ y_{..} &= 371.8 & \bar{y}_{..} &= 17.70 & N &= 21 \end{aligned}$$

216

One-way ANOVA

Fish length example

- The sums of squares are:

$$\begin{aligned} SS_{\text{Total}} &= \sum_{i=1}^3 \sum_{j=1}^7 y_{ij}^2 - \frac{(y_{..})^2}{N} \\ &= 6618.60 - 6582.63 = 35.97 \end{aligned}$$

$$\begin{aligned} SS_{\text{Trt}} &= \sum_{i=1}^3 \frac{(y_{i.})^2}{n_i} - \frac{(y_{..})^2}{N} \\ &= \frac{1}{7} [(130.3)^2 + (123.6)^2 + (117.9)^2] - 6582.63 \\ &= 11.01 \end{aligned}$$

$$SS_{\text{Err}} = SS_{\text{Tot}} - SS_{\text{Trt}} = 35.97 - 11.01 = 24.96$$

- Or $SS_{\text{Err}} = 6s_1^2 + 6s_2^2 + 6s_3^2 = 24.96$
- The corresponding ANOVA table is:

Source	df	SS	MS
Trt	2	11.01	5.505
Error	18	24.96	1.387
Total	20	35.97	

217

One-way ANOVA

Fish length example

- Note that the MS for Error computed above is the same as the pooled estimate of variance, s_p^2 .
- The null hypothesis H_0 : "all population means are equal" versus the alternative hypothesis H_A : "not all population means are equal".

- The observed test statistic is:

$$f = \frac{MST_{\text{Trt}}}{MSE_{\text{Err}}} = \frac{5.505}{1.387} = 3.97$$

- Compare this with $F_{2,18}$ from Table D: at 5% $f_{2,18} = 3.55$, and at 1% $f_{2,18} = 6.01$, so for our data $0.01 < \text{p-value} < 0.05$.
- Reject H_0 at the 5% level. There is moderate evidence against H_0 . That is, there is moderate evidence that there is a diet effect on the fish length.

218

One-way ANOVA

Assumptions

- For each trt, a random sample $Y_{ij} \sim N(\mu_i, \sigma_i^2)$.
- Equal variances $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2$.
- Independent samples across trts.

That is, independence, normality, and equal variances.

A unified model

$$Y_{ij} = \mu_i + e_{ij}$$

where e_{ij} are iid $N(0, \sigma^2)$. Let

$$\mu = \frac{1}{k} \sum_{i=1}^k \mu_i, \quad \alpha_i = \mu_i - \mu.$$

Then equivalently the model is:

$$Y_{ij} = \mu + \alpha_i + e_{ij}$$

where e_{ij} are iid $N(0, \sigma^2)$.

219

One-way ANOVA

Hypotheses

$H_0 : \mu_1 = \mu_2 = \dots = \mu_k$ vs. H_A : Not all μ_i 's are equal.

Equivalently

$H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ vs. H_A : Not all α_i 's are zero.

F-test

Under H_0 , the test statistic is

$$F = \frac{MST_{\text{Trt}}}{MSE_{\text{Error}}} \sim F_{\text{dfTrt}, \text{dfError}}$$

Parameter estimation

- Estimate σ^2 by S_p^2 .
- Estimate μ_i by \bar{Y}_i .
- Or estimate μ by $\bar{Y}_{..}$ and estimate α_i by $\bar{Y}_i - \bar{Y}_{..}$.
- We will discuss inference of parameters later on.

220

One-way ANOVA

A brief review

Dist'n	One-Sample Inference	Two-Sample Inference
Normal	$H_0: \mu = \mu_0$	Paired $H_0: \mu_D = 0$, CI for μ_D (Z or T_{n-1})
	CI for μ	2 ind samples $H_0: \mu_1 = \mu_2$, CI for $\mu_1 - \mu_2$ ($T_{n_1+n_2-2}$)
	σ^2 is known (Z) or unknown (T_{n-1})	k ind samples $H_0: \mu_1 = \mu_2 = \dots = \mu_k$ ($F_{k-1, N-k}$)
	$H_0: \sigma^2 = \sigma_0^2$, CI for σ^2 (χ_{n-1}^2)	$H_0: \sigma^2 = \sigma_0^2$, CI for σ^2 (χ_{N-k}^2)
Arbitrary	$H_0: \mu = \mu_0$, CI for μ (CLT Z)	Paired $H_0: \mu_D = 0$ (Signed rank)
		2 ind samples $H_0: \mu_1 = \mu_2$ (Mann-Whitney)
		2 ind samples $H_0: \sigma_1^2 = \sigma_2^2$ (Levene's)
Binomial	$H_0: p = p_0$ (Binomial $Y \sim B(n, p)$)	
	$H_0: p = p_0$, CI for p (CLT Z)	2 ind samples $H_0: p_1 = p_2$, CI for $p_1 - p_2$ (CLT Z)

- For testing or CI, address model assumptions (e.g. normality, independence, equal variance) via detection, correction, and robustness.
- In hypothesis testing, H_0, H_A (1-sided or 2-sided), test statistic and its distribution, p-value, interpretation, rejection region, α, β , power, sample size determination.
- For paired t-test, the assumptions are $D \sim \text{iid } N(\mu_D, \sigma_D^2)$ where $D = Y_1 - Y_2$. Y_1, Y_2 need not be normal. Y_1 and Y_2 need not be independent.

One-way ANOVA

More on assumptions

Assumptions	Detection
Normality	Stem-and-leaf plot; normal scores plot
Independence	Study design
Equal variance	Levene's test
Correct model	More later

Detect unequal variance

- Plot trt standard deviation vs trt mean.
- Or use an extension of Levene's test for

$$H_0: \sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2.$$

The main idea remains the same, except that a one-way ANOVA is used instead of a two-sample t-test.

One-way ANOVA

Levene's test

For example, consider $k = 3$ groups of data.

Sample 1: 2, 5, 7, 10
 Sample 2: 4, 8, 19
 Sample 3: 1, 2, 4, 4, 7

- (1) Find the median for each sample. Here $\tilde{y}_1 = 6, \tilde{y}_2 = 8, \tilde{y}_3 = 4$.
- (2) Subtract the median from each obs and take absolute values.

Sample 1*: 4, 1, 1, 4
 Sample 2*: 4, 0, 11
 Sample 3*: 3, 2, 0, 0, 3

- (3) For any sample that has an odd sample size, remove 1 zero.

Sample 1*: 4, 1, 1, 4
 Sample 2*: 4, 11
 Sample 3*: 3, 2, 0, 3

- (4) Perform a one-way ANOVA f-test on the final results.

Source	df	SS	MS	F	p-value
Group	2	44.6	22.30	3.95	$0.05 < p < 0.10$
Error	7	39.5	5.64	-	
Total	9	84.1	-	-	

One-way ANOVA

Key R commands

```
> # Fish length example
> y1 = c(18.2, 20.1, 17.6, 16.8, 18.8, 19.7, 19.1)
> y2 = c(17.4, 18.7, 19.1, 16.4, 15.9, 18.4, 17.7)
> y3 = c(15.2, 18.8, 17.7, 16.5, 15.9, 17.1, 16.7)
> y = c(y1, y2, y3)
> n1 = length(y1)
> n2 = length(y2)
> n3 = length(y3)
> trt = c(rep(1, n1), rep(2, n2), rep(3, n3))
> oneway.test(y~factor(trt), var.equal=T)

One-way analysis of means

data: y and factor(trt)
F = 3.9683, num df = 2, denom df = 18, p-value = 0.03735

> fit.lm = lm(y~factor(trt))
> anova(fit.lm)
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value Pr(>F)
factor(trt) 2 11.0067  5.5033  3.9683 0.03735 *
Residuals 18 24.9629  1.3868
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> # Alternatively use data frame
> eg = data.frame(y=y, trt=factor(trt))
> eg
  y trt
1 18.2 1
2 20.1 1
3 17.6 1
4 16.8 1
5 18.8 1
6 19.7 1
```

```

7 19.1 1
8 17.4 2
9 18.7 2
10 19.1 2
11 16.4 2
12 15.9 2
13 18.4 2
14 17.7 2
15 15.2 3
16 18.8 3
17 17.7 3
18 16.5 3
19 15.9 3
20 17.1 3
21 16.7 3
> eg.lm = lm(y~trt, eg)
> anova(eg.lm)
Analysis of Variance Table

Response: y
Df Sum Sq Mean Sq F value Pr(>F)
trt      2 11.0067    5.5033   3.9683 0.03735 *
Residuals 18 24.9629    1.3868
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
> # Kruskal-Wallis rank sum test
> kruskal.test(y~trt)

      Kruskal-Wallis rank sum test

data:  y by trt
Kruskal-Wallis chi-squared = 5.7645, df = 2, p-value = 0.05601

```

225

Comparisons among Means

An overview

- In one-way ANOVA, if we reject H_0 , then we know that not all trt means are the same.
- But this may not be informative enough. We now consider particular comparisons of trt means.
- We will consider contrasts and all pairwise comparisons.

226

Comparisons among Means

Fish length example continued

Recall the example with $k = 3$ trts and $n = 7$ obs/trt. Test $H_0 : \mu_1 = \mu_3$ vs $H_A : \mu_1 \neq \mu_3$.

- $\bar{y}_1 = 18.61, \bar{y}_3 = 16.84, n_1 = n_3 = 7, s_p = 1.387$ on $df = 18$.
- The observed test statistic is

$$t = \frac{\bar{y}_1 - \bar{y}_3}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_3}}} = \frac{18.61 - 16.84}{\sqrt{1.387 \times \frac{2}{7}}} = 2.81$$

on $df = 18$. The p-value $2 \times P(T_{18} \geq 2.81)$ is between 0.01 and 0.02.

- We may also construct a $(1 - \alpha)$ CI for $\mu_1 - \mu_3$:

$$(\bar{y}_1 - \bar{y}_3) \pm t_{df, \alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_3}}$$

- Suppose $\alpha = 0.05$. Thus $t_{18, 0.025} = 2.101$ and a 95% CI for $\mu_1 - \mu_3$ is

$$(18.61 - 16.84) \pm 2.101 \times \sqrt{1.387 \times \frac{2}{7}}$$

which is $[0.45, 3.09]$ or 1.77 ± 1.32 .

227

Comparisons among Means

Fish length example continued

Now test $H_0 : \mu_1 - \frac{1}{2}(\mu_2 + \mu_3) = 0$ vs $H_A : \mu_1 - \frac{1}{2}(\mu_2 + \mu_3) \neq 0$.

- Estimate $\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$ by $\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3)$.
- The test statistic is

$$T = \frac{\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3) - \mu_{\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3)}}{S_{\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3)}}$$

- We will see that

$$\mu_{\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3)} = \mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$$

and

$$S_{\bar{Y}_1 - \frac{1}{2}(\bar{Y}_2 + \bar{Y}_3)} = s_p \sqrt{\frac{1}{n_1} + \frac{1}{4n_2} + \frac{1}{4n_3}}$$

- Thus a $(1 - \alpha)$ CI for $\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$ is

$$\bar{y}_1 - \frac{1}{2}(\bar{y}_2 + \bar{y}_3) \pm s_p \sqrt{\frac{1}{n_1} + \frac{1}{4n_2} + \frac{1}{4n_3}}$$

- But first we will generalize this situation.

228

Comparisons among Means

Contrast

- A *contrast* is a quantity of the form

$$\sum_{i=1}^k \lambda_i \mu_i$$

where k is the # of trts, μ_i is the i^{th} trt mean, and λ_i is the i^{th} contrast coefficient.

- For comparison, we require that $\sum_{i=1}^k \lambda_i = 0$.
- For example, we have seen two contrasts already.
- $\mu_1 - \mu_3$ is a contrast with $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$:

$$\sum_{i=1}^k \lambda_i \mu_i = 1 \times \mu_1 + 0 \times \mu_2 + (-1) \times \mu_3.$$

- $\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$ is a contrast with $\lambda_1 = 1, \lambda_2 = -1/2, \lambda_3 = -1/2$:

$$\sum_{i=1}^k \lambda_i \mu_i = 1 \times \mu_1 + (-1/2) \times \mu_2 + (-1/2) \times \mu_3.$$

229

Comparisons among Means

Contrast

- Estimate $\sum_{i=1}^k \lambda_i \mu_i$ by $X = \sum_{i=1}^k \lambda_i \bar{Y}_i$.
- Consider the distribution of

$$T = \frac{X - \mu_X}{S_X}$$

- Here $\mu_X = \sum_{i=1}^k \lambda_i \mu_i$, because

$$\mu_X = E\left(\sum_{i=1}^k \lambda_i \bar{Y}_i\right) = \sum_{i=1}^k \lambda_i E(\bar{Y}_i) = \sum_{i=1}^k \lambda_i \mu_i.$$

- For S_X , consider variance first.

$$\text{Var}\left(\sum_{i=1}^k \lambda_i \bar{Y}_i\right) = \sum_{i=1}^k \lambda_i^2 \text{Var}(\bar{Y}_i) = \sum_{i=1}^k \lambda_i^2 \frac{\sigma^2}{n_i} = \sigma^2 \sum_{i=1}^k \frac{\lambda_i^2}{n_i}.$$

- Estimate $\text{Var}(\sum_{i=1}^k \lambda_i \bar{Y}_i)$ by $S_p^2 \sum_{i=1}^k \frac{\lambda_i^2}{n_i}$ and

$$S_X = S_p \sqrt{\sum_{i=1}^k \frac{\lambda_i^2}{n_i}}$$

230

Comparisons among Means

Fish length example continued

- For the first contrast, $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$,

$$S_X = S_p \sqrt{\frac{1}{7} + \frac{0}{7} + \frac{1}{7}} = S_p \sqrt{\frac{2}{7}}$$

as before.

- For the second contrast, $\lambda_1 = 1, \lambda_2 = -1/2, \lambda_3 = -1/2$,

$$S_X = S_p \sqrt{\frac{1}{7} + \frac{1/4}{7} + \frac{1/4}{7}} = S_p \sqrt{\frac{3}{14}}$$

- Thus for testing $H_0 : \mu_1 - \frac{1}{2}(\mu_2 + \mu_3) = 0$, the observed test statistic is

$$t = \frac{\bar{y}_1 - \frac{1}{2}(\bar{y}_2 + \bar{y}_3)}{s_p \sqrt{\frac{3}{14}}} = \frac{18.61 - (17.66 + 16.84)/2}{\sqrt{1.387 \times \frac{3}{14}}} = 2.49$$

on $df = 18$. The p-value $2 \times P(T_{18} \geq 2.49)$ is between 0.02 and 0.05.

231

Comparisons among Means

Fish length example continued

- We may also construct a 95% CI for $\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$:

$$\bar{y}_1 - \frac{1}{2}(\bar{y}_2 + \bar{y}_3) \pm t_{18,0.025} s_p \sqrt{\frac{3}{14}}$$

- A 95% CI for $\mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$ is

$$18.61 - \frac{1}{2}(17.66 + 16.84) \pm 2.101 \times \sqrt{1.387 \times \frac{3}{14}}$$

which is $[0.21, 2.51]$ or 1.36 ± 1.15 .

232

Comparisons among Means

Remarks

- If all $n_i = n$, then

$$\text{Var}\left(\sum_{i=1}^k \lambda_i \bar{Y}_i\right) = \frac{\sigma^2}{n} \sum_{i=1}^k \lambda_i^2.$$

This is called a *balanced* case.

- Single sample $S_{\bar{Y}} = S \sqrt{\frac{1}{n}}$
- Two samples $S_{\bar{Y}_1 - \bar{Y}_2} = S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$
- Multiple samples

$$S_{\sum_{i=1}^k \lambda_i \bar{Y}_i} = S_p \sqrt{\sum_{i=1}^k \frac{\lambda_i^2}{n_i}}$$