Confidence Intervals

An overview

- Most probability distributions are indexed by one or more parameters.
- For example, $N(\mu, \sigma^2)$ or $B(n, p)$.
- In significance tests, we have used point estimators for parameters.
- For example, for iid $Y_1, Y_2, \ldots, Y_n \sim N(\mu, \sigma^2)$, $\bar{Y}$ is a point estimator of $\mu$ and $S^2$ is a point estimator of $\sigma^2$.
- Note that $E(\bar{Y}) = \mu$ and $E(S^2) = \sigma^2$. That is, $\bar{Y}$ is an unbiased estimator of $\mu$ and $S^2$ is an unbiased estimator of $\sigma^2$.
- Another example, for $Y \sim B(n, p)$, $\hat{p} = Y/n$ is an unbiased (point) estimator of $p$, because $E(\hat{p}) = p$.
- Now we study interval estimator to give a reasonable interval for parameters (e.g. $(c_1, c_2)$ for $\mu$).
- The assumptions are the same as in significance testing, but we do not need a null hypothesis on the parameters (e.g. $\mu = \mu_0$).

Confidence Intervals

Normal distribution with known $\sigma^2$

- Suppose $Y_1, Y_2, \ldots, Y_n$ are iid from $N(\mu, \sigma^2)$ and $\sigma^2$ is known.
- We know that $\bar{Y}$ estimates $\mu$, but $\bar{Y}$ can be off somewhat.
- Our goal is to get a plausible range of values for $\mu$ based on the sample data.
- Recall that $\bar{Y} \sim N(\mu, \sigma^2/n)$ Hence
  
  $$0.95 = P(-1.96 \leq Z \leq 1.96) = P(-1.96 \leq \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \leq 1.96) = P(-1.96\frac{\sigma}{\sqrt{n}} \leq \bar{Y} - \mu \leq 1.96\frac{\sigma}{\sqrt{n}}) = P(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}})$$

- Note that $\bar{Y}$ is random and $\mu$ is fixed.
- The interval
  
  $$\bar{y} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{y} + 1.96\frac{\sigma}{\sqrt{n}}$$

  is called a 95% confidence interval for $\mu$ (or 0.95 CI for $\mu$).

Confidence Intervals

Remarks

- It is not true that $P(8.23 \leq \mu \leq 13.77) = 0.95$ because once a sample is observed, there is nothing random.
- The 95% probability has to do with the procedure. It is interpreted as, 95% of the time, the CI’s calculated in this way contain $\mu$.
- For a single case, it is interpreted as having 95% confidence that $\mu$ is between 8.23 and 13.77.
- The interval $[8.23, 13.77]$ can be thought of as a plausible range of $\mu$.

Confidence Intervals

Normal CI example

Suppose there are eight (8) observations in a sample from $N(\mu, 16)$ and the observed sample mean is $\bar{y} = 11.00$. Then $n = 8$, $\sigma^2 = 16$, and a 95% CI for $\mu$ is

$$11.00 - 1.96\frac{4}{\sqrt{8}} \leq \mu \leq 11.00 + 1.96\frac{4}{\sqrt{8}}$$

which is

$$8.23 \leq \mu \leq 13.77$$

or

$$11.00 \pm 2.77$$
Confidence Intervals

Remarks

• In general, let \( z_{\alpha/2} \) denote the \( z \)-score such that
  \[ P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha. \]

Then we have
  \[ 1 - \alpha = P(Y - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq Y + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}) \]

• A 100(1 - \(\alpha\))% confidence interval for \( \mu \) (or (1 - \(\alpha\)) CI for \( \mu \)) is
  \[ \bar{y} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{y} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]
  or
  \[ \bar{y} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]

Normal CI example continued

Continued with the CI example that has \( \bar{y} = 11.00, n = 8, \sigma^2 = 16. \) Find a 90% CI for \( \mu \).

• Since 1 - \(\alpha\) = 0.90, we have \(\alpha = 0.10, \alpha/2 = 0.05, z_{\alpha/2} = 1.645 \) (using Table A or the table on page 92).

• Then a 90% CI for \( \mu \) is
  \[ 11.00 - 1.645 \frac{4}{\sqrt{8}} \leq \mu \leq 11.00 + 1.645 \frac{4}{\sqrt{8}} \]
  which is
  \[ 8.67 \leq \mu \leq 13.33 \]
  or
  \[ 11.00 \pm 2.33 \]

• By convention, CI’s are two-sided. But one-sided confidence bounds are possible.

Confidence Intervals

Normal distribution with unknown \(\sigma^2\)

• Suppose \( Y_1, Y_2, \ldots, Y_n \) are iid from \( N(\mu, \sigma^2) \) and \(\sigma^2\) is unknown.

• Recall that
  \[ \frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim T_{n-1} \]

• Let \( t_{\alpha/2} \) denote the \(t\)-score such that
  \[ P(-t_{n-1, \alpha/2} \leq T_{n-1} \leq t_{n-1, \alpha/2}) = 1 - \alpha. \]

Then we have
  \[ 1 - \alpha = P(Y - t_{n-1, \alpha/2} \frac{S}{\sqrt{n}} \leq \mu \leq Y + t_{n-1, \alpha/2} \frac{S}{\sqrt{n}}) \]

• A (1 - \(\alpha\)) CI for \( \mu \) is
  \[ \bar{y} - t_{n-1, \alpha/2} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{y} + t_{n-1, \alpha/2} \frac{S}{\sqrt{n}} \]
  or
  \[ \bar{y} \pm t_{n-1, \alpha/2} \frac{S}{\sqrt{n}} \]

Tomato weight example

• Recall a random sample of \( n = 16 \) tomatoes that has a sample mean weight of \( \bar{y} = 32.50 \) gm.

• Previous we assumed that the weight of tomatoes have a normal distribution \( N(\mu, (5)^2) \).

• Thus a 95% CI for \( \mu \) is
  \[ 32.50 - 1.96 \times \frac{5}{\sqrt{16}} \leq \mu \leq 32.50 + 1.96 \times \frac{5}{\sqrt{16}} \]
  which is \([30.05, 34.95]\) or \(32.50 \pm 2.45\).

• But suppose we do not know what \(\sigma\) is. Compute a sample variance which turns out to \(s^2 = 30.02\).

• Then since 1 - \(\alpha\) = 0.95, we have \(\alpha = 0.05, \alpha/2 = 0.025, t_{n-1, \alpha/2} = t_{15, 0.025} = 2.131 \) (using Table C), and \(s/\sqrt{n} = \sqrt{30.02/16} = 1.370\)

• Then a 95% CI for \( \mu \) is
  \[ 32.50 - 2.131 \times 1.370 \leq \mu \leq 32.50 + 2.131 \times 1.370 \]
  which is \([29.58, 35.42]\) or \(32.50 \pm 2.92\).
Confidence Intervals

Inference for unspecified distribution

- Suppose we have a large number of observations from an unspecified distribution with mean \( \mu \) and variance \( \sigma^2 \). Also suppose that \( \sigma^2 \) is unknown. Our goal is to construct a CI for \( \mu \).
- By the CLT and \( S^2 \approx \sigma^2 \) when \( n \) is large, we have the fact that
  \[ \frac{\bar{y} - \mu}{S/\sqrt{n}} \approx N(0, 1) \]
- Thus an approximate \((1 - \alpha)\) CI for \( \mu \) is
  \[ \bar{y} - \frac{s}{\sqrt{n}} \leq \mu \leq \bar{y} + \frac{s}{\sqrt{n}} \]
- For example, suppose a random sample of size \( n = 85 \), the observed sample mean and sample variance are \( \bar{y} = 25.50, s^2 = 8.3 \).
- For a 95% CI, \( z_{\alpha/2} = 1.96 \) and
  \[ 25.50 - 1.96 \times \sqrt{\frac{8.3}{85}} \leq \mu \leq 25.50 + 1.96 \times \sqrt{\frac{8.3}{85}} \]
  which is [25.14, 25.86] or 25.50 ± 0.36.

Confidence Intervals

Remarks

- For example, suppose a random sample of size \( n = 8 \) from \( N(\mu, \sigma^2) \) with unknown \( \sigma^2 \). Suppose the observed sample and sample variance are \( \bar{y} = 37.1, S^2 = 6.29 \).
- For testing \( H_0 : \mu = 35 \) versus \( H_A : \mu \neq 35 \), we use t-test. The observed t-score is
  \[ t = \frac{\bar{y} - \mu_0}{s/\sqrt{n}} = \frac{37.1 - 35}{\sqrt{6.29}/8} = 2.368 \]
  and \( t_{2.025} = 2.365 \) (using Table C). Thus barely reject \( H_0 \) at the 5% level.
- A 95% CI for \( \mu \) is
  \[ 37.1 - 2.365\sqrt{6.29}/8 \leq \mu \leq 37.1 + 2.365\sqrt{6.29}/8 \]
  which is [35.003, 39.197], which does not contain 35 (but barely).

Confidence Intervals

Inference for binomial distribution

- Now consider \( Y \sim B(n, p) \) and construct a \((1 - \alpha)\) CI for \( p \) using normal approximation.
- Recall that \( \hat{p} = Y/n \) is a point estimator of \( p \) with
  \[ E(\hat{p}) = p, \ Var(\hat{p}) = \frac{pq}{n} \]
- Also recall that when \( np \geq 5, nq \geq 5 \), we can approximate the distribution of \( \hat{p} \) by
  \[ \hat{p}_{NA} \sim N(p, \frac{pq}{n}) \]
- For testing \( H_0 : p = p_0 \), use
  \[ \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \approx N(0, 1) \]
  Now since \( \hat{p} \approx p \),
  \[ \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \approx N(0, 1) \]
- Thus an approximate \((1 - \alpha)\) CI for \( p \) is
  \[ \hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \]
Confidence Intervals

Binomial CI example

In an experiment, a drug is given to treat 200 rats with a certain disease and 63 of them are cured. Let \( p \) denote the cure rate. Let \( Y \) denote the number of rats cured and assume that \( Y \sim B(n, p) \). Construct a 95% CI for \( p \).

- Here \( n = 200, y = 63 \), and the observed \( \hat{p} = y/n = 63/200 = 0.315 \).
- Since \( n\hat{p} = 63 > 5, n(1-\hat{p}) = 137 > 5 \), we use normal approximation.
- Since \( z_{\alpha/2} = Z_{0.025} = 1.96 \), a 95% CI for \( p \) is
  
  \[
  0.315 - 1.96 \times \sqrt{0.315 \times 0.685/200} \leq p \leq 0.315 + 1.96 \times \sqrt{0.315 \times 0.685/200}
  \]
  
  which is \([0.251, 0.379]\) or \(0.315 \pm 0.064\).
- Normal approximation is appropriate if \( n\hat{p} \geq 5 \).

The relation between significance testing and CI's for \( p \) is not exact, because different variance terms are used.

Confidence Intervals

A quick summary

<table>
<thead>
<tr>
<th>( Y_1, \ldots, Y_n )</th>
<th>( n )</th>
<th>( \sigma ) known</th>
<th>( \sigma ) unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(\mu, \sigma^2) ) small</td>
<td>( \bar{y} \pm \frac{z_{\alpha/2}}{\sqrt{n}} \sigma )</td>
<td>( \bar{y} \pm \frac{t_{n-1, \alpha/2}}{\sqrt{n}} \sigma )</td>
<td></td>
</tr>
<tr>
<td>( N(\mu, \sigma^2) ) large</td>
<td>( \bar{y} \pm \frac{z_{\alpha/2}}{\sqrt{n}} \sigma )</td>
<td>( \bar{y} \pm \frac{t_{n-1, \alpha/2}}{\sqrt{n}} \sigma )</td>
<td></td>
</tr>
<tr>
<td>( D(\mu, \sigma^2) ) small</td>
<td>no general result</td>
<td>no general result</td>
<td></td>
</tr>
<tr>
<td>( D(\mu, \sigma^2) ) large</td>
<td>( \bar{y} \pm \frac{z_{\alpha/2}}{\sqrt{n}} \sigma ) by CLT</td>
<td>( \bar{y} \pm \frac{z_{\alpha/2}}{\sqrt{n}} \sigma ) by CLT</td>
<td></td>
</tr>
</tbody>
</table>

Confidence Intervals

Key R commands

```
# Normal CI example
ybar = 11
d = 4
alpha = 0.05
z = qnorm(alpha/2, lower.tail=F)
z*d/sqrt(n)
[1] 2.771808

# Tomato weight example
ybar = 32.5
n = 16
alpha = 0.05
sd = 5
z = qnorm(alpha/2, lower.tail=F)
z*sd/sqrt(n)
[1] 2.449955

# Fruit can example
sd = sqrt(30.02)
sd/sqrt(n)
[1] 1.369763
t = qt(alpha/2, n-1, lower.tail=F)
t*sd/sqrt(n)
[1] 2.131450

# CLT CI example
ybar = 25.5
sd = 0.5
alpha = 0.05
z = qnorm(alpha/2, lower.tail=F)
zd/sqrt(n)
[1] 0.3576283
```

Confidence Intervals

Key R commands

```
# Tomato weight example
ybar = 32.5
n = 16
alpha = 0.05
sd = 5
z = qnorm(alpha/2, lower.tail=F)
zd/sqrt(n)
[1] 2.449955

# Fruit can example
sd = sqrt(30.02)
zd/sqrt(n)
[1] 1.369763
t = qt(alpha/2, n-1, lower.tail=F)
tzd
[1] 2.131450

# CLT CI example
ybar = 25.5
sd = 0.5
alpha = 0.05
zd/sqrt(n)
[1] 0.3576283
```
Confidence Intervals

Key R commands

> # Binomial CI example
> y=63
> n=200
> phat = y/n
> alpha = 0.05
> z = qnorm(alpha/2, lower.tail=F)
> sqrt(phat*(1-phat)/n)
[1] 0.06437743
> c(phat-z*sqrt(phat*(1-phat)/n),phat+z*sqrt(phat*(1-phat)/n))
[1] 0.2506226 0.3793774
> # or, directly
> prop.test(y, n, conf.level=0.95)

1-sample proportions test with continuity correction

data: y out of n, null probability 0.5
X-squared = 26.645, df = 1, p-value = 2.445e-07
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
0.2523053 0.3849353
sample estimates:

p
0.315