Consider a simple gene A with alleles a_1 and a_2. The three possible genotypes are a_1a_1, a_1a_2, and a_2a_2. The probability distribution of the genotype of a single offspring for each possible cross are as follows.

	Offspring Genotype		
Parent Genotypes	a_1a_1	a_1a_2	a_2a_2
$a_1a_1 \times a_1a_1$	1	0	0
$a_1a_1 \times a_1a_2$	1/2	1/2	0
$a_1a_1 \times a_2a_2$	0	1	0
$a_1a_2 \times a_1a_2$	1/4	1/2	1/4
$a_1a_2 \times a_2a_2$	0	1/2	1/2
$a_2a_2 \times a_2a_2$	0	0	1

Suppose that there is a trait T that is partially determined by the single gene A and partially determined by environmental conditions. Specifically, suppose that the probability that an individual exhibits trait T depends on the genotype as follows: $\mathbb{P}(T \mid a_1a_1) = 0.6$, $\mathbb{P}(T \mid a_1a_2) = 0.3$, and $\mathbb{P}(T \mid a_2a_2) = 0.05$. Furthermore, suppose that given the genotype information in an individual, no further information regarding genotypes or trait incidence in others affect the probability that the individual has the trait. (Traits are conditionally independent given the genotypes among a group of individuals.)

In a certain population, the proportions of individuals with genotypes a_1a_1, a_1a_2, and a_2a_2 are respectively 0.5, 0.4, and 0.1. Now consider a cross between PARENT 1, selected at random from the population, with PARENT 2, an individual with genotype a_1a_1. Suppose that the cross produces a single offspring, OFFSPRING.

In solving each of the following problems, present sufficient work so that your method of solution is clear. Clearly define any notation for events you introduce for the problem. Please use the space below and on the back of the page to show your exam work. Express all answers as decimals with at least three significant figures.

(a) What is the probability that PARENT 1 has the trait?

(b) Given that PARENT 1 has the trait, what is the probability that PARENT 1 has genotype a_1a_2?

(c) If PARENT 1 has genotype a_1a_2, what is the probability that OFFSPRING has the trait?

(d) What is the probability that OFFSPRING has the trait?

(e) Given that OFFSPRING has the trait, what is the probability that PARENT 1 has genotype a_1a_2?

Solution: It is important to define events associated with the problem. Here, we have the genotype of Parent 1, the genotype of the Offspring, and whether or not Parent 1 and Offspring have the trait T. Genotypes are either a_1a_1, a_1a_2, or a_2a_2. Using notation for events that should be self-explanatory, the given information is $\mathbb{P}(P_1 = a_1a_1) = 0.5$, $\mathbb{P}(P_1 = a_1a_2) = 0.4$, $\mathbb{P}(P_1 = a_2a_2) = 0.1$, $\mathbb{P}(P_2 = a_1a_1) = 1$. For any individual, $\mathbb{P}(T \mid a_1a_1) = 0.6$, $\mathbb{P}(T \mid a_1a_2) = 0.3$, and $\mathbb{P}(T \mid a_2a_2) = 0.05$.

The following solutions describe how to use equations to find the answers to each problem. In class, I will demonstrate a probability tree approach to setting the problems up.

(a) Find $\mathbb{P}(P_1 \text{ has } T)$ using the law of total probability (sum the probabilities of paths through the tree).

$$
\begin{aligned}
\mathbb{P}(P_1 \text{ has } T) &= \mathbb{P}(P_1 \text{ has } T \mid P_1 = a_1a_1)\,\mathbb{P}(P_1 = a_1a_1) \\
&\quad + \mathbb{P}(P_1 \text{ has } T \mid P_1 = a_1a_2)\,\mathbb{P}(P_1 = a_1a_2) \\
&\quad + \mathbb{P}(P_1 \text{ has } T \mid P_1 = a_2a_2)\,\mathbb{P}(P_1 = a_2a_2) \\
&= (0.6)(0.5) + (0.3)(0.4) + (0.05)(0.1) \\
&= 0.425.
\end{aligned}
$$

(b) Find $\mathbb{P}(P_1 = a_1a_2 \mid P_1 \text{ has } T)$, first by the definition of conditional probability and ultimately using Bayes' theorem. This is equivalent to renormalizing the probabilities of the remaining paths through the tree so that they again sum to one.

$$\begin{aligned}\mathbb{P}(P_1 = a_1a_2 \mid P_1 \text{ has } T) &= \frac{\mathbb{P}(P_1 = a_1a_2 \text{ and } P_1 \text{ has } T)}{\mathbb{P}(P_1 \text{ has } T)} \\ &= \frac{(0.3)(0.4)}{0.425} \\ &\doteq 0.282\end{aligned}$$

(c) If $P_1 = a_1a_2$, then either $O = a_1a_1$ or $O = a_1a_2$ with probability 0.5 each as $P_2 = a_1a_1$ and the offspring gets an a_1 from P_2 for certain. Thus, by the conditional law of total probability, we have the following for $\mathbb{P}(O \text{ has } T \mid P_1 = a_1a_2)$.

$$\begin{aligned}\mathbb{P}(O \text{ has } T \mid P_1 = a_1a_2) &= \mathbb{P}(O \text{ has } T \mid P_1 = a_1a_2 \text{ and } O = a_1a_1)\,\mathbb{P}(O = a_1a_1 \mid P_1 = a_1a_2) \\ &\quad + \mathbb{P}(O \text{ has } T \mid P_1 = a_1a_2 \text{ and } O = a_1a_2)\,\mathbb{P}(O = a_1a_2 \mid P_1 = a_1a_2) \\ &= (0.6)(0.5) + (0.3)(0.5) \\ &= 0.45\end{aligned}$$

(d) Now, we do not condition on the genotype of Parent 1 to find $\mathbb{P}(O \text{ has } T)$. Application of the law of total probability a couple times leads to the following.

$$\begin{aligned}\mathbb{P}(O \text{ has } T) &= \mathbb{P}(O \text{ has } T \mid O = a_1a_1)\,\mathbb{P}(O = a_1a_1) + \mathbb{P}(O \text{ has } T \mid O = a_1a_2)\,\mathbb{P}(O = a_1a_2) \\ &= (0.6)((0.5)(1) + (0.4)(0.5) + (0.1)(0)) + (0.3)((0.5)(0) + (0.4)(0.5) + (0.1)(1)) \\ &= 0.51\end{aligned}$$

(e) The last problem uses Bayes' Theorem to find $\mathbb{P}(P_1 = a_1a_2 \mid O \text{ has } T)$.

$$\begin{aligned}\mathbb{P}(P_1 = a_1a_2 \mid O \text{ has } T) &= \frac{\mathbb{P}(P_1 = a_1a_2 \text{ and } O \text{ has } T)}{\mathbb{P}(O \text{ has } T)} \\ &= \frac{(0.4)(0.5)(0.6) + (0.4)(0.5)(0.3)}{0.51} \\ &\doteq 0.353\end{aligned}$$