Consider the following genetics setting. A gene has two alleles, A and a. The A allele is completely dominant so that every individual with the heterozygous genotype Aa expresses the dominant phenotype as do all dominant homozygotes AA, while each homozygous recessive individual (aa) expresses the recessive phenotype.

A cross of AA and aa individuals produces the F_1 generation, all of whom are heterozygous (Aa). The F_2 generation is formed by crossing individuals from the F_1 generation. The probabilities of genotypes AA, Aa, and aa are 1/4, 1/2, and 1/4, respectively. Consider a cross between an F_2 individual with the dominant phenotype (so this individual's genotype is either AA or Aa) and an individual from the F_1 generation.

Please express your answers to the following questions as fractions, not as decimals.

(a) What is the probability that the F_2 parent is heterozygous for the gene given that this parent expresses the dominant phenotype?

(b) If the $\mathrm{F}_1 \times$ dominant F_2 cross produces one offspring, what is the probability that this offspring has the dominant phenotype?

(c) Assume that phenotypes of offspring are independent of one another given the genotypes of the parents. If the $\mathrm{F}_1 \times$ dominant F_2 cross produces two offspring, what is the probability that both offspring have the dominant phenotype?

(d) Given that both offspring have the dominant phenotype, what is the probability that the dominant F_2 parent is a heterozygote?

Solution:

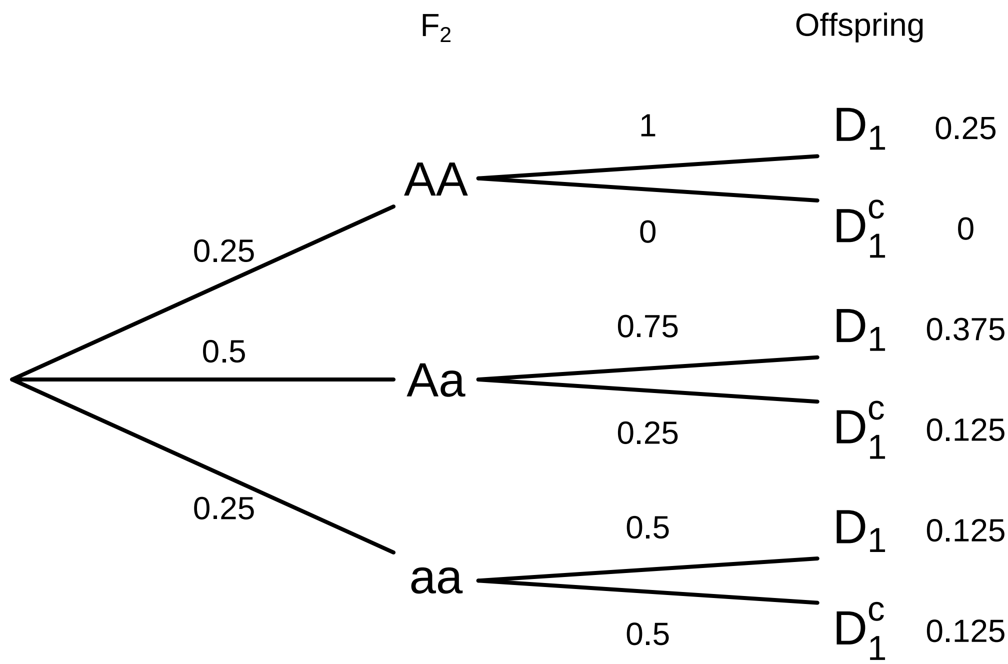
(a) Let $\mathrm{F}_2 = Aa$ be the event that the F_2 parent is a heterozygote and let $\mathrm{F}_2 = AA$ be the event that the F_2 parent is homozygous dominant. Then, if $\mathrm{F}_2 = D$ is the event that this parent has the dominant phenotype, $(\mathrm{F}_2 = D) = (\mathrm{F}_2 = AA)$ or $(\mathrm{F}_2 = Aa)$. Notice that $\Pr\{\mathrm{F}_2 = D\} = \Pr\{\mathrm{F}_2 = AA\} + \Pr\{\mathrm{F}_2 = Aa\} = 3/4$. We then have

$$\Pr\{\mathrm{F}_2 = Aa \,|\, \mathrm{F}_2 = D\} = \frac{\Pr\{\mathrm{F}_2 = Aa \text{ and } \mathrm{F}_2 = D\}}{\Pr\{\mathrm{F}_2 = D\}} = \frac{1/2}{3/4} = \frac{2}{3}$$

(b) Let D_1 be the event that the one offspring has the dominant phenotype. We are asked to find $\Pr\{D_1 \,|\, \mathrm{F}_2 = D\}$. One solution conditions on the genotype of the F_2 parent.

$$\begin{aligned}
\Pr\{D_1 \,|\, \mathrm{F}_2 = D\} &= \Pr\{\mathrm{F}_2 = AA \,|\, \mathrm{F}_2 = D\}\Pr\{D_1 \,|\, \mathrm{F}_2 = AA, \mathrm{F}_2 = D\} + \Pr\{\mathrm{F}_2 = Aa \,|\, \mathrm{F}_2 = D\}\Pr\{D_1 \,|\, \mathrm{F}_2 = Aa, \mathrm{F}_2 = D\} \\
&= \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right) \\
&= \frac{5}{6}
\end{aligned}$$

Here is an alternative solution that uses a full probability tree.

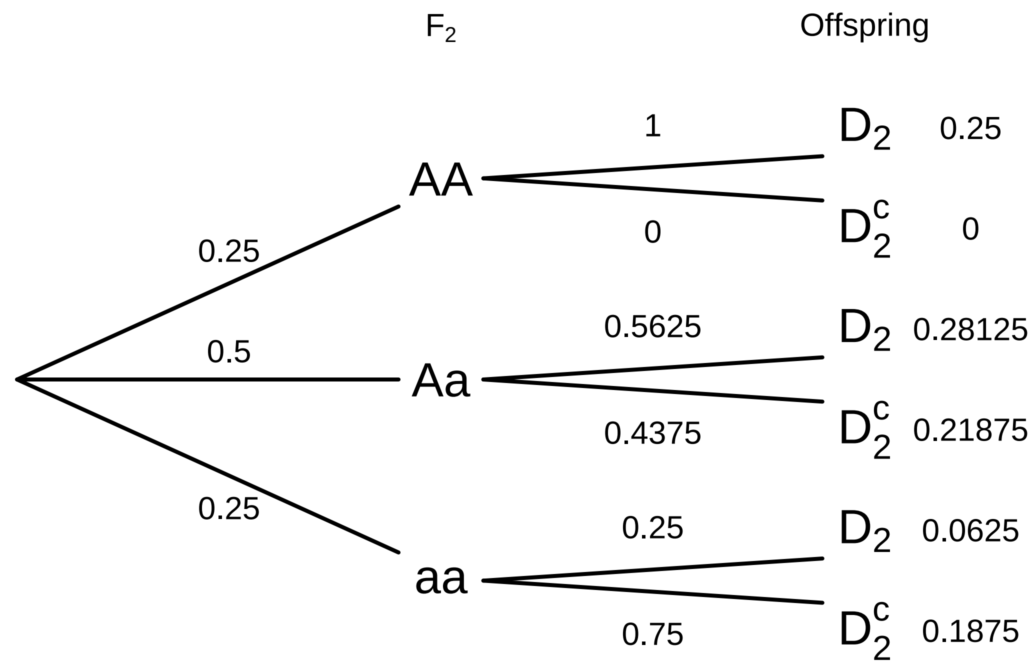


$$
\begin{aligned}
\Pr\{D_1 \mid \mathrm{F}_2 = D\} &= \frac{\Pr\{D_1 \text{ and } \mathrm{F}_2 = D\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{\Pr\{D_1 \text{ and } \mathrm{F}_2 = AA\} + \Pr\{D_1 \text{ and } \mathrm{F}_2 = Aa\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{\Pr\{\mathrm{F}_2 = AA\}\Pr\{D_1 \mid \mathrm{F}_2 = AA\} + \Pr\{\mathrm{F}_2 = Aa\}\Pr\{D_1 \mid \mathrm{F}_2 = Aa\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{(1/4)(1) + (1/2)(3/4)}{3/4} \\
&= \frac{5/8}{3/4} \\
&= \frac{5}{6}
\end{aligned}
$$

(c) Let D_2 be the event that both offspring are dominant. One solution using equations and conditionaing on the dominance of the F_2 parent is as follows.

$$
\begin{aligned}
\Pr\{D_2 \mid \mathrm{F}_2 = D\} &= \Pr\{\mathrm{F}_2 = AA \mid \mathrm{F}_2 = D\}\Pr\{D_2 \mid \mathrm{F}_2 = AA, \mathrm{F}_2 = D\} + \Pr\{\mathrm{F}_2 = Aa \mid \mathrm{F}_2 = D\}\Pr\{D_2 \mid \mathrm{F}_2 = Aa, \mathrm{F}_2 = D\} \\
&= \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2 \\
&= \frac{8}{24} + \frac{9}{24} \\
&= \frac{17}{24}
\end{aligned}
$$

We could also set up a tree much like the first.



$$
\begin{aligned}
\Pr\{D_2 \mid \mathrm{F}_2 = D\} &= \frac{\Pr\{D_2 \text{ and } \mathrm{F}_2 = D\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{\Pr\{D_2 \text{ and } \mathrm{F}_2 = AA\} + \Pr\{D_2 \text{ and } \mathrm{F}_2 = Aa\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{\Pr\{\mathrm{F}_2 = AA\}\Pr\{D_2 \mid \mathrm{F}_2 = AA\} + \Pr\{\mathrm{F}_2 = Aa\}\Pr\{D_2 \mid \mathrm{F}_2 = Aa\}}{\Pr\{\mathrm{F}_2 = D\}} \\
&= \frac{(1/4)(1) + (1/2)(3/4)^2}{3/4} \\
&= \frac{17/32}{3/4} \\
&= \frac{17}{24}
\end{aligned}
$$

(d) Use Bayes' Theorem and the defintion of conditional probability.

$$
\begin{aligned}
\Pr\{\mathrm{F}_2 = Aa \mid D_2, \mathrm{F}_2 = D\} &= \frac{\Pr\{\mathrm{F}_2 = Aa \text{ and } D_2 \mid \mathrm{F}_2 = D\}}{\Pr\{D_2 \mid \mathrm{F}_2 = D\}} \\
&= \frac{9/24}{17/24} \\
&= \frac{9}{17}
\end{aligned}
$$