The Normal Theory Distributions

Introduction The basic distributions of normal theory can be defined in terms of random variables. Let $Z_1, Z_2, ..., Z_n$ be n independent standard normal random variables. Then

$$\chi^2_m = \sum_{i=1}^{m} Z_i^2 \quad \text{has a chi square distribution with } m \text{ d. f.}$$

If a standard normal variable Z and chi square variable χ^2_m are independent, then

$$t_m = \frac{Z}{\sqrt{\chi^2_m / m}} \qquad \text{has a t distribution with } m \text{ d. f.}$$

Then, if the chi square variable χ^2_m and the chi square variable χ^2_n are independent

$$F_{m,n} = \frac{\chi^2_m / m}{\chi^2_n / n} \qquad \text{has an F distribution with } (m, n) \text{ d. f.}$$

The basic case arises starting with n independent normal random variables $X_1, X_2, ..., X_n$ all having the same mean μ and standard deviation σ. Then

$$Z_i = \frac{X_i - \mu}{\sigma}$$

has a standard normal distribution for each i. It then holds that

$$\sqrt{n}\ \overline{Z} = \sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n} Z_i = \sqrt{n}\,(\frac{\overline{X} - \mu)}{\sigma})$$

has a standard normal distribution.

Next

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} (Z_i - \overline{Z} + \overline{Z})^2 = \sum_{i=1}^{n} (Z_i - \overline{Z})^2 + n\overline{Z}^2$$

The left hand side has a chi-square distribution with n degrees of freedom. The last terms on the right is the square of the standard normal variable $\sqrt{n}\,\overline{Z}$ and so has a chi-square distribution with 1 degree of freedom. It can be shown that the two terms on the right-hand side of the equation are independent and that

$$\sum_{i=1}^{n} (Z_i - \overline{Z})^2 \qquad \text{has a chi-square distribution with } \; n - 1 \; \text{ degrees of freedom}$$

Since

$$\frac{(n - 1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2} = \sum_{i=1}^{n} (Z_i - \overline{Z})^2$$

we conclude that $(n - 1)S^2/\sigma^2$ has a chi-square distribution with $n - 1$ degrees of freedom.