Name: ______________________________

1. (4 points) If $s = (x_1, x_2, \ldots, x_n)$ is an i.i.d. sample from a Binomial$(2, \theta)$ distribution, find the maximum likelihood estimator of θ.

   Solution: The likelihood is

$$\prod_{i=1}^{n} \binom{2}{x_i} \theta^{x_i} (1-\theta)^{2-x_i} = \left(\prod_{i=1}^{n} \binom{2}{x_i} \right) \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{2n - \sum_{i=1}^{n} x_i} = \left(\prod_{i=1}^{n} \binom{2}{x_i} \right) \theta^{n\bar{x}} (1-\theta)^{n(2-\bar{x})}$$

   The log likelihood is

$$\log \left(\prod_{i=1}^{n} \binom{2}{x_i} \right) + n\bar{x} \log\theta + n(2-\bar{x}) \log(1-\theta).$$

   The derivative of the log-likelihood set to zero is the equation

$$\frac{n\bar{x}}{\theta} - \frac{n(2-\bar{x})}{1-\theta} = 0$$

   which has solution

$$\hat{\theta} = \frac{n\bar{x}}{n\bar{x} + n(2-\bar{x})} = \frac{\bar{x}}{2}.$$

   This is a maximum since the second derivative evaluated at $\hat{\theta}$ is

$$-\frac{n\bar{x}}{\theta^2} - \frac{n(2-\bar{x})}{(1-\theta)^2} < 0$$

   when $0 < \hat{\theta} < 2$. When $\bar{x} = 0$, the likelihood is $(1-\theta)^{2n}$ which decreases, so the maximum is $\hat{\theta} = 0$ and when $\bar{x} = 2$, the likelihood is θ^{2n} which increases and achieves its maximum at $\hat{\theta} = 1$. Thus, $\hat{\theta} = \bar{x}/2$ is the maximum likelihood estimator for all samples.

2. Assume that $X_i \sim$ i.i.d. Binomial$(2, \theta_1)$ for $i = 1, \ldots, 10$ and that $Y_i \sim$ i.i.d. Binomial$(2, \theta_2)$ for $i = 1, \ldots, 10$ and that the samples $s_1 = (x_1, x_2, \ldots, x_{10})$ and $s_2 = (y_1, y_2, \ldots, y_{10})$ are independent of each other. The parameter space is $\Omega = \{\theta = (\theta_1, \theta_2) : 0 < \theta_1, \theta_2 < 1\}$. The observed data is $s_1 = (0, 0, 0, 0, 0, 1, 1, 1, 1, 0)$ and $s_2 = (0, 1, 0, 1, 1, 1, 2, 2, 1, 1)$.

   (a) (5 points) Write an expression for the log-likelihood $\log f_\theta(s)$ under the full model in terms of θ_1 and θ_2 for the data $s = (s_1, s_2)$. Use the result from problem 1 to give numerical estimates for the maximum likelihood estimates of θ_1 and θ_2 and for $\max_{\theta \in \Omega} \log f_\theta(s)$.

   Solution:

   For two independent samples, the likelihood is the product of the likelihoods for each sample and the log-likelihood is the sum of the two log-likelihoods. We can plug in to the expression above, noting that $\bar{x} = 0.4$

so $\hat\theta_1 = 0.4/2 = 0.2$ and $\bar y = 1$ so $\hat\theta_2 = 1/2 = 0.5$. The log-likelihood is the following.

$$\begin{aligned}
\log f_\theta(s) &= \log f_{\theta_1}(s_1) + \log f_{\theta_2}(s_2) \\
&= \left(\log\left(\prod_{i=1}^{10}\binom{2}{x_i}\right) + 10\bar x\log\theta_1 + 10(2-\bar x)\log(1-\theta_1)\right) \\
&\quad + \left(\log\left(\prod_{i=1}^{10}\binom{2}{y_i}\right) + 10\bar y\log\theta_2 + 10(2-\bar y)\log(1-\theta_2)\right) \\
&= 4\log(2) + 4\log(0.2) + 16\log(0.8) + 6\log(2) + 10\log(0.5) + 10\log(0.5) \\
&\doteq -16.9395
\end{aligned}$$

(b) (5 points) A null model assumes that $\theta_1 = \theta_2$: let γ be this common value. Write an expression for the log-likelihood $\log f_\gamma(s)$ under this null model in terms of γ. For this null model, use the result from problem 1 to give numerical estimates for the maximum likelihood estimate for γ and for $\max_{\theta\in\Omega_0}\log f_\theta(s)$ where $\Omega_0 = \{\theta = (\theta_1,\theta_2) : 0 < \theta_1,\theta_2 < 1, \theta_1 = \theta_2\}$.

Solution: For the null model, we can combine the two samples into one, which has a combined sample mean of $14/20 = 0.7$. The estimated parameter is $\hat\gamma = 0.7/2 = 0.35$. The log-likelihood is then

$$\begin{aligned}
\log f_\gamma(s) &= \log\left(\prod_{i=1}^{20}\binom{2}{s_i}\right) + 20\bar s\log\gamma + 20(2-\bar s)\log(1-\gamma) \\
&= 10\log 2 + 14\log(0.35) + 26\log(0.65) \doteq -18.9664
\end{aligned}$$

(c) (4 points) Find a numerical value for the test statistic for the likelihood ratio test of null hypothesis $H_0\colon \theta_1 = \theta_2$ versus alternative $H_A\colon \theta_1 \neq \theta_2$.

Solution: The test statistic is $-2\log\Lambda = -2(-18.9664 - -16.9395) = 4.0537$.

(d) (2 points) Circle choices or fill in the blanks. The p-value for this hypothesis test is equal to the area to the RIGHT of 4.0537 below the density of a chi-square distribution with 1 degree of freedom.