Sample $s = (x_1, x_2, \ldots, x_n)$ is drawn independently from a Poisson(θ) model, conditional on the value of θ. Recall the Poisson probability function $\mathsf{P}(X_i = x \mid \theta) = \mathrm{e}^{-\theta}\theta^x/x!$ for $x = 0, 1, 2, \ldots$ for all i, and that $\mathsf{E}(X \mid \theta) = \theta$.

1. Assume the Bayesian prior distribution $\theta \sim$ Exponential(2). Recall that the Exponential(λ) density is $f(t) = \lambda \mathrm{e}^{-\lambda t}$ with mean $1/\lambda$.

   (a) (2 points) Find a numerical value for $\mathsf{E}(X_i)$, the prior predictive mean.
   Solution:
   $$\mathsf{E}(X_i) = \mathsf{E}(\mathsf{E}(X_i \mid \theta)) = \mathsf{E}(\theta) = 1/2$$

   (b) (2 points) Calculate $\mathsf{P}(X_1 = 1)$. (*Hint: You need to average over possible values of θ. Recall either the Gamma function or the Gamma density.*)
   Solution:
   $$\begin{aligned} \mathsf{P}(X_1 = 1) &= \int_0^\infty \left(\frac{\mathrm{e}^{-\theta}\theta^1}{1!}\right)\left(2\mathrm{e}^{-2\theta}\right) \mathrm{d}\theta \\ &= \frac{2\Gamma(2)}{3^2}\int_0^\infty \frac{3^2}{\Gamma(2)}\theta^{2-1}\mathrm{e}^{-3\theta}\,\mathrm{d}\theta \\ &= 2/9 \end{aligned}$$
   The integral expression simplifies to one as it is a gamma density integrated over its range.

   (c) (6 points) Suppose that sample s is summarized by $n = 50$ and $\bar{x} = 0.96$. Find an equation for the posterior density $\pi(\theta \mid s)$.
   Solution: The posterior density is proportional to the likelihood times the prior density. Note that $\sum_{i=1}^{50} x_i = 50(0.96) = 48$. The likelihood is $\mathrm{e}^{-50\theta}\theta^{48}/\prod_{i=1}^{50} x_i!$.
   $$\begin{aligned} \pi(\theta \mid s) &\propto \left(\mathrm{e}^{-50\theta}\theta^{48}\right)\left(\mathrm{e}^{-2\theta}\right) \\ &= \theta^{49-1}\mathrm{e}^{-52\theta} \end{aligned}$$
   which implies that $\theta \mid s \sim$ Gamma$(49, 52)$ so that
   $$\pi(\theta \mid s) = \frac{52^{49}}{\Gamma(49)}\theta^{49-1}\mathrm{e}^{-52\theta} \quad \text{for } 0 < \theta < \infty$$

   (d) (4 points) Fill in the blanks.
   Solution: If s is the sample from the previous part, a 90 percent credible region for θ of the form (a, b) can be found where a and b are equal to the 0.05 and 0.95 quantiles, respectively, from the Gamma distribution with parameters $\alpha = 49$ and $\lambda = 52$.

2. (6 points) Assume the same likelihood model and a sample s where $n = 50$ and $\bar{x} = 0.96$. Use a prior distribution where θ has possible values 1/3 and 1 with prior probabilities $\pi(1/3) = 3/4$ and $\pi(1) = 1/4$. Calculate $\pi(1/3 \mid s)$ to the nearest six decimal places.

Solution:

$$
\begin{aligned}
\pi(1/3 \,|\, s) &= \frac{(3/4)\mathrm{e}^{-50/3}(1/3)^{48} / \prod_{i=1}^{50} x_i!}{(3/4)\mathrm{e}^{-50/3}(1/3)^{48} / \prod_{i=1}^{50} x_i! + (1/4)\mathrm{e}^{-50}(1)^{48} / \prod_{i=1}^{50} x_i!} \\
&= \frac{3\mathrm{e}^{-50/3}(1/3)^{48}}{3\mathrm{e}^{-50/3}(1/3)^{48} + \mathrm{e}^{-50}} \\
&= \frac{3\mathrm{e}^{100/3}(1/3)^{48}}{3\mathrm{e}^{100/3}(1/3)^{48} + 1} \\
&\approx 1.12664e-08 \\
&\doteq 0.000000
\end{aligned}
$$