A likelihood model for sample $s = (1, 3, 2, 1, 0)$ is that $X_i \sim$ i.i.d. Geometric(θ). Specifically, the probability function is $\mathsf{P}(X_i = x \mid \theta) = \theta(1-\theta)^x$ for $x = 0, 1, 2, \ldots$ for all i. Two different Bayesian analyses use separate prior distributions for θ.

1. One person assumes that $\theta \sim \text{Beta}(\alpha_0, \beta_0)$, so that $\pi(\theta) = \frac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)}\theta^{\alpha_0-1}(1-\theta)^{\beta_0-1}$. Recall that the mean of the Beta(α, β) distribution is $\alpha/(\alpha+\beta)$.

   (a) (6 points) Let $\alpha_0 = \beta_0 = 2$. Find the posterior density $\pi(\theta \mid s)$. (No need for integration!)

   Solution: The posterior density is proportional to the product of the prior density and the likelihood.

$$\begin{aligned}\pi(\theta \mid s) &\propto \pi(\theta) f_\theta(s) \\ &\propto \theta^{\alpha_0-1}(1-\theta)^{\beta_0-1}\theta^n(1-\theta)^{\sum_{i=1}^n x_i} \\ &= \theta^{n+\alpha_0-1}(1-\theta)^{\sum_{i=1}^n x_i+\beta_0-1}\end{aligned}$$

   It follows that $\theta \mid s \sim \mathsf{Beta}(n+\alpha_0, \sum_{i=1}^n x_i + \beta_0)$. Here, $n = 5$, $\alpha_0 = \beta_0 = 2$, and $\sum_{i=1}^n x_i = 7$, so $\theta \mid s \sim \mathsf{Beta}(7, 9)$ and the posterior density is

$$\pi(\theta \mid s) = \frac{\Gamma(16)}{\Gamma(7)\Gamma(9)}\theta^6(1-\theta)^8, \quad 0 < \theta < 1.$$

   (b) (6 points) Find numerical quantities for the posterior mean and mode (maximum value).

   Solution: The posterior mean is the mean of the corresponding Beta distribution which is $7/(7+9) = 0.4375$. The posterior mode is found by finding the maximum of the posterior density.

$$\frac{\partial}{\partial\theta}\left(6\log\theta + 8\log(1-\theta)\right) = \frac{6}{\theta} - \frac{8}{1-\theta} = 0$$

   which implies

$$6 - 6\theta = 8\theta$$

   and the mode is $\theta = 6/14 = 3/8 \doteq 0.375$.

   (c) (2 points) Write down (but do not evaluate) an integral expression equal to the posterior probability $\mathsf{P}(\theta > 0.5 \mid s)$.

   Solution: The posterior probability is the integral of the posterior density from 0.5 to 1.

$$\mathsf{P}(\theta > 0.5 \mid s) = \int_{0.5}^1 \frac{\Gamma(16)}{\Gamma(7)\Gamma(9)}\theta^6(1-\theta)^8 \, \mathrm{d}\theta$$

2. (6 points) A second person has a discrete prior probability distribution for θ where $\pi(w) = \mathsf{P}(\theta = w)$; $\pi(1/3) = 0.2$, $\pi(1/2) = 0.3$ and $\pi(2/3) = 0.5$. Find the posterior distribution $\pi(\theta \mid s)$ for the sample above.

Solution:

The posterior probability function is proportional to the product of the likelihood and the prior probability function. Here, the posterior distribution will be restricted to the three points 1/3, 1/2, and 2/3. A table can help organize the calculation. The total of the column headed $\pi(\theta)f_\theta(s)$ is 0.0001515 and the answer is found by dividing each separate value here by this total.

θ	$\pi(\theta)$	$f_\theta(s)$	$\pi(\theta)f_\theta(s)$	$\pi(\theta \mid s)$
1/3	0.2	$(1/3)^5(2/3)^7$	4.817e-05	0.3179
1/2	0.3	$(1/2)^5(1/2)^7$	7.324e-05	0.4834
2/3	0.5	$(2/3)^5(1/3)^7$	3.011e-05	0.1987