An i.i.d. sample $X_1, \ldots, X_n$ is taken from an absolutely continuous distribution with density $f_\theta(x) = \frac{1}{\theta^2} e^{-x/\theta}$ for $x > 0$ where $\theta \in \Omega = (0, \infty)$ is an unknown parameter. For this distribution, $E(X_i) = 2\theta$ and $\text{Var}(X_i) = 2\theta^2$. Recall that for sufficiently large samples that the distribution of the maximum likelihood estimate (MLE) $\hat{\theta}$ is approximately $N(\theta, (nI(\theta))^{-1})$. A sample with $n = 100$ has the following summary statistics: $\sum_{i=1}^{n} x_i = 191.8$, $\sum_{i=1}^{n} \log x_i = 35.4$, $\sum_{i=1}^{n} x_i^2 = 556.0$, $\min_i \{x_i\} = 0.13$, and $\max_i \{x_i\} = 6.52$. This table has some quantiles of a standard normal distribution, $\Phi(z) = P(Z < z)$.

<table>
<thead>
<tr>
<th>$z$</th>
<th>0.8</th>
<th>0.9</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi(z)$</td>
<td>0.8416</td>
<td>1.2816</td>
<td>1.6449</td>
<td>1.9600</td>
<td>2.3263</td>
<td>2.5758</td>
<td>3.0902</td>
</tr>
</tbody>
</table>

1. (6 points) Find an expression for the maximum likelihood estimate $\hat{\theta}$ of $\theta$ and evaluate it numerically.

Solution: Find the log-likelihood, take its derivative and equate to zero, solve, and verify the second derivative is negative at this solution.

$$f_\theta(X) = \prod_{i=1}^{n} x_i^{1/\theta} e^{-x_i/\theta} = \theta^{-2n} e^{-n\bar{x}/\theta} \left( \prod_{i=1}^{n} x_i \right)$$

$$\log f_\theta(s) = -2n \log \theta - \frac{n\bar{x}}{\theta} + \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial \log f_\theta(s)}{\partial \theta} = \frac{-2n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0$$

$$\hat{\theta} = \frac{\bar{x}}{2} \approx 0.959$$

The second derivative is negative at $\hat{\theta}$.

$$\frac{\partial^2 \log f_\theta(s)}{\partial \theta^2} \bigg|_{\theta=\hat{\theta}} = \frac{2n}{\theta^2} - \frac{2n\bar{x}}{\theta^3} \bigg|_{\theta=\hat{\theta}} = \frac{8n}{\bar{x}^2} - \frac{16n}{\bar{x}^2} = -\frac{8n}{\bar{x}^2} < 0$$

2. (6 points) Find an expression for the Fisher information $I(\theta)$ for this distribution and find a numerical estimate of it.

Solution: We find $I(\theta)$ by calculating the expected value of the negative of the second derivative of the log likelihood of an observation.

$$f_\theta(X) = \frac{X}{\theta^2} e^{-X/\theta}$$

$$\log f_\theta(X) = -2 \log \theta - \frac{X}{\theta} + \log X$$

$$\frac{\partial \log f_\theta(X)}{\partial \theta} = \frac{-2}{\theta} + \frac{X}{\theta^2}$$

$$-\frac{\partial^2 \log f_\theta(X)}{\partial \theta^2} = -\frac{2}{\theta^2} + \frac{2X}{\theta^3}$$

$$I(\theta) = -\frac{2}{\theta^2} + \frac{2E(X)}{\theta^3}$$

$$= -\frac{2}{\theta^2} + \frac{2(2\theta)}{\theta^3}$$

$$= \frac{2}{\theta^2}$$

$$I(\hat{\theta}) = 2/(0.959)^2 \approx 2.175.$$  Also, $nI(\hat{\theta}) = 200/(0.959)^2 \approx 217.5$. 
3. (4 points) Construct a 95% confidence interval for $\theta$ using the large sample properties of the MLE.

Solution: Begin by recalling $\hat{\theta} \approx N(\theta, (nI(\theta))^{-1})$ and so we are 95% confident that the unknown $\theta$ is in the interval

$$\hat{\theta} \pm 1.96 \frac{1}{\sqrt{nI(\hat{\theta})}} \text{ or } 0.959 \pm 0.133 \text{ or } (0.826, 1.092)$$

4. (4 points) In a test of the null hypothesis $H_0: \theta = 1$ versus the two-sided alternative $H_A: \theta \neq 1$, find the value of the test statistic and determine if the p-value is greater than or lower than 0.05.

Solution: State hypotheses, find the test statistic, and compute a p-value.

$H_0: \theta = 1$, $H_A: \theta \neq 1$.

$$Z = \frac{\hat{\theta} - \theta_0}{1/\sqrt{100} \times 2/\theta_0^2} = \sqrt{200}(0.959 - 1) \approx -0.58$$

Since $| -0.58 | < 1.96$, the p-value is greater than 0.05.