An i.i.d. sample $X_1, \ldots, X_n$ is taken from an absolutely continuous distribution with density $f_\theta(x) = \frac{x}{\theta^2} e^{-x/\theta}$ for $x > 0$ where $\theta \in \Omega = (0, \infty)$ is an unknown parameter. For this distribution, $\mathsf{E}(X_i) = 2\theta$ and $\mathsf{Var}(X_i) = 2\theta^2$. Recall that for sufficiently large samples that the distribution of the maximum likelihood estimate (MLE) $\hat\theta$ is approximately $\mathrm{N}(\theta, (nI(\theta))^{-1})$. A sample with $n = 100$ has the following summary statistics: $\sum_{i=1}^n x_i = 191.8$, $\sum_{i=1}^n \log x_i = 35.4$, $\sum_{i=1}^n x_i^2 = 556.0$, $\min_i\{x_i\} = 0.13$, and $\max_i\{x_i\} = 6.52$. This table has some quantiles of a standard normal distribution, $\Phi(z) = \mathsf{P}(Z < z)$.

z	0.8	0.9	0.95	0.975	0.99	0.995	0.999
$\Phi(z)$	0.8416	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

1. (6 points) Find an expression for the maximum likelihood estimate $\hat\theta$ of θ and evaluate it numerically.

Solution: Find the log-likelihood, take its derivative and equate to zero, solve, and verify the second derivative is negative at this solution.

$$\begin{aligned}
f_\theta(s) &= \prod_{i=1}^n \frac{x_i}{\theta^2} e^{-x_i/\theta} = \theta^{-2n} e^{-n\bar{x}/\theta} \left(\prod_{i=1}^n x_i \right) \\
\log f_\theta(s) &= -2n \log\theta - \frac{n\bar{x}}{\theta} + \sum_{i=1}^n \log x_i \\
\frac{\partial \log f_\theta(s)}{\partial\theta} &= \frac{-2n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0 \\
\hat\theta &= \frac{\bar{x}}{2} \doteq 0.959
\end{aligned}$$

The second derivative is negative at $\hat\theta$.

$$\frac{\partial^2 \log f_\theta(s)}{\partial\theta^2}\Big|_{\theta=\hat\theta} = \frac{2n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}\Big|_{\theta=\hat\theta} = \frac{8n}{\bar{x}^2} - \frac{16n}{\bar{x}^2} = -\frac{8n}{\bar{x}^2} < 0$$

2. (6 points) Find an expression for the Fisher information $I(\theta)$ for this distribution and find a numerical estimate of it.

Solution: We find $I(\theta)$ by calculating the expected value of the negative of the second derivative of the log likelihood of an observation.

$$\begin{aligned}
f_\theta(X) &= \frac{X}{\theta^2} e^{-X/\theta} \\
\log f_\theta(X) &= -2\log\theta - \frac{X}{\theta} + \log X \\
\frac{\partial \log f_\theta(X)}{\partial\theta} &= \frac{-2}{\theta} + \frac{X}{\theta^2} \\
-\frac{\partial^2 \log f_\theta(X)}{\partial\theta^2} &= -\frac{2}{\theta^2} + \frac{2X}{\theta^3} \\
I(\theta) &= -\frac{2}{\theta^2} + \frac{2\mathsf{E}(X)}{\theta^3} \\
&= -\frac{2}{\theta^2} + \frac{2(2\theta)}{\theta^3} \\
&= \frac{2}{\theta^2}
\end{aligned}$$

$I(\hat\theta) = 2/(0.959)^2 \doteq 2.175$. Also, $nI(\hat\theta) = 200/(0.959)^2 \doteq 217.5$.

3. (4 points) Construct a 95% confidence interval for θ using the large sample properties of the MLE.

   Solution: Begin by recalling $\hat{\theta} \approx \mathsf{N}(\theta, (nI(\theta))^{-1})$ and so we are 95% confident that the unknown θ is in the interval

$$\hat{\theta} \pm 1.96 \frac{1}{\sqrt{nI(\hat{\theta})}} \quad \text{or} \quad 0.959 \pm 0.133 \quad \text{or} \quad (0.826, 1.092)$$

4. (4 points) In a test of the null hypothesis $H_0 : \theta = 1$ versus the two-sided alternative $H_A : \theta \neq 1$, find the value of the test statistic and determine if the p-value is greater than or lower than 0.05.

   Solution: State hypotheses, find the test statistic, and compute a p-value.

   $H_0 : \theta = 1$, $H_A : \theta \neq 1$.

$$Z = \frac{\hat{\theta} - \theta_0}{1/\sqrt{100 \times 2/\theta_0^2}} = \sqrt{200}(0.959 - 1) \doteq -0.58$$

   Since $|-0.58| < 1.96$, the p-value is greater than 0.05.