An i.i.d. sample $X_1, \ldots, X_n$ is taken from a distribution F with density $f_\theta(x) = (\theta^3/2)x^{-4}\mathrm{e}^{-\theta/x}$ for $x > 0$ where $\theta \in \Omega = (0, \infty)$ is an unknown parameter. For this distribution, $\mathsf{E}(X) = \theta/2$ and $\mathsf{Var}(X) = \theta^2/4$.

1. (6 points) Find the maximum likelihood estimate (MLE) $\hat{\theta}$ of θ.

   Solution:

$$\begin{aligned}
L(\theta \mid s) &= \prod_{i=1}^n \frac{\theta^3}{2} x_i^{-4} \mathrm{e}^{-\theta/x_i} \\
&= \theta^{3n} 2^{-n} \left(\prod_{i=1}^n x_i^{-4} \right) \mathrm{e}^{-\theta \sum_{i=1}^n x_i^{-1}} \\
\ell(\theta \mid s) = \log L(\theta \mid s) &= 3n \log \theta - n \log 2 - 4 \left(\sum_{i=1}^n \log x_i \right) - \theta \sum_{i=1}^n x_i^{-1} \\
\frac{\partial \ell(\theta \mid s)}{\partial \theta} &= \frac{3n}{\theta} - \sum_{i=1}^n x_i^{-1} = 0 \\
\hat{\theta} &= \frac{3n}{\sum_{i=1}^n x_i^{-1}}
\end{aligned}$$

   Note that the second partial derivative is $\frac{\partial^2 \ell(\theta)}{\partial \theta^2} = -3n\theta^{-2} < 0$, so $\hat{\theta}$ is a maximum of the likelihood.

2. (6 points) For sample $s = (x_1, \ldots, x_n)$, show that $T(s) = \sum_{i=1}^n x_i^{-1}$ is a minimal sufficient statistic.

   Solution: We can write the likelihood as

$$L(\theta \mid s) = \left(\theta^{3n} \mathrm{e}^{-\theta \sum_{i=1}^n x_i^{-1}} \right) \left(2^{-n} \left(\prod_{i=1}^n x_i^{-4} \right) \right)$$

   so by the factorization theorem, $T(s) = \sum_{i=1}^n x_i^{-1}$ is a sufficient statistic. To show it is a minimal sufficient statistic, let $s_1 = (x_1, \ldots, x_n)$ and $s_2 = (y_1, \ldots, y_n)$ be two samples. If the likelihood ratio $L(\theta \mid s_1)/L(\theta \mid s_2)$ does not depend on θ, then

$$\begin{aligned}
\frac{L(\theta \mid s_1)}{L(\theta \mid s_2)} &= \frac{\left(\mathrm{e}^{-\theta \sum_{i=1}^n x_i^{-1}} \theta^{3n} \right) \left(2^{-n} \left(\prod_{i=1}^n x_i^{-4} \right) \right)}{\left(\mathrm{e}^{-\theta \sum_{i=1}^n y_i^{-1}} \theta^{3n} \right) \left(2^{-n} \left(\prod_{i=1}^n y_i^{-4} \right) \right)} \\
&= \mathrm{e}^{-\theta \left(\sum_{i=1}^n x_i^{-1} - \sum_{i=1}^n y_i^{-1} \right)} \left(\prod_{i=1}^n (y_i/x_i)^4 \right)
\end{aligned}$$

   does not depend on θ which implies that $\sum_{i=1}^n x_i^{-1} - \sum_{i=1}^n y_i^{-1} = 0$, or that $T(s_1) = T(s_2)$. Thus $T(s)$ is a minimal sufficient statistic.

3. (5 points) An alternative estimate of θ is $\tilde{\theta} = 2\bar{x}$. Find the bias, variance, and mean square error of $\tilde{\theta}$.

   Solution:

$$\mathsf{E}(\tilde{\theta}) = \mathsf{E}(2\bar{X}) = \frac{2}{n} \sum_{i=1}^n \mathsf{E}(X_i) = \left(\frac{2}{n} \right) \left(\frac{n\theta}{2} \right) = \theta$$

$$\mathsf{Var}(\tilde{\theta}) = \mathsf{Var}(2\bar{X}) = \frac{4}{n^2}\sum_{i=1}^{n}\mathsf{Var}(X_i) = \left(\frac{4}{n^2}\right)\left(\frac{n\theta^2}{4}\right) = \frac{\theta^2}{n}$$

As the bias is 0, the MSE is the variance,

$$\mathsf{MSE}(\tilde{\theta}) = \frac{\theta^2}{n}$$

4. (2 points) It is a fact that $Y = \sum_{i=1}^{n} X_i^{-1} \sim \text{Gamma}(3n, \theta)$. The corresponding density is

$$f_Y(y) = \frac{\theta^{3n}}{(3n-1)!} y^{3n-1} \mathrm{e}^{-\theta y}$$

for $y > 0$. Find the bias of the MLE.

Solution: Since the MLE is $3n/\sum_{i=1}^{n} x_i^{-1}$ which has the same distribution as $3n/Y$, where $Y \sim \mathsf{Gamma}(3n, \theta)$, the bias is equal to $\mathsf{E}(3n/Y) - \theta$.

$$\begin{aligned}
\mathsf{E}\left(\frac{3n}{Y}\right) &= 3n\int_0^\infty \frac{1}{y}\frac{\theta^{3n}}{(3n-1)!} y^{3n-1}\mathrm{e}^{-\theta y}\mathrm{d}y \\
&= 3n\frac{\theta^{3n}}{(3n-1)!}\frac{(3n-2)!}{\theta^{3n-1}}\int_0^\infty \frac{\theta^{3n-1}}{(3n-2)!} y^{(3n-1)-1}\mathrm{e}^{-\theta y}\mathrm{d}y \\
&= \frac{3n\theta}{3n-1}
\end{aligned}$$

So the bias is

$$\frac{3n\theta}{3n-1} - \theta = \frac{\theta}{3n-1}\ .$$

5. (1 point) You may assume that $\mathsf{Var}(\hat{\theta}) = 9n^2\theta^2/((3n-1)^2(3n-2))$. Find an expression for $\text{MSE}(\tilde{\theta})/\text{MSE}(\hat{\theta})$ in terms of n. If $n = 10$, does $\tilde{\theta}$ or $\hat{\theta}$ have a smaller mean square error?

Solution: Since the MSE is the bias squared plus the variance,

$$\mathsf{MSE}(\hat{\theta}) = 9n^2\theta^2/((3n-1)^2(3n-2)) + \left(\frac{\theta}{3n-1}\right)^2 = \frac{\theta^2(9n^2+3n-2)}{(3n-1)^2(3n-2)} = \frac{\theta^2(3n+2)}{(3n-1)(3n-2)}\ .$$

The ratio is

$$\frac{\mathsf{MSE}(\tilde{\theta})}{\mathsf{MSE}(\hat{\theta})} = \frac{\theta^2/n}{\theta^2(3n+2)/((3n-1)(3n-2))} = \frac{(3n-1)(3n-2)}{n(3n+2)}$$

which is equal to $\frac{812}{320}$ when $n = 10$. Despite having a bias, the MLE has a much smaller MSE than the unbiased estimate $\tilde{\theta}$.