A new method for measuring information about the physical conformation of large organic molecules records the times that photons of a specific wavelength hit a detector. At a particular experimental setup, the expected rate that photons hit the detector is 5.3 photons per microsecond (10^{-6} seconds). Assume that arrivals of photons at the detector are modeled as a Poisson process. Let N_t be the number of photons detected up to time t where t is measured in microseconds.

Recall that if $X \sim \text{Poisson}(\mu)$, then $\mathsf{P}(X = k) = \mathrm{e}^{-\mu}\mu^k/k!$ for $k = 0, 1, 2, \ldots$ and $\mathsf{E}(X) = \mathsf{Var}(X) = \mu$. In addition, interarrival times for a Poisson process at rate θ are independent exponential random variables with density $\theta\mathrm{e}^{-\theta t}$ for $t > 0$.

1. (5 points) What is the probability that exactly four photons are detected within the first 1.5 microseconds?

   Solution: Since $5.3 \times 1.5 = 7.95$, $N_{1.5} \sim \text{Poisson}(7.95)$. $\mathsf{P}(N_{1.5} = 4) = \mathrm{e}^{-7.95}(7.95)^4/4! \doteq 0.0587$.

2. (5 points) Find $\mathsf{P}(N_2 = 10 \mid N_{1.5} = 4)$.

   Solution: There must be 6 arrivals in the time interval $(1.5, 2)$, and the mean of the distribution is $5.3 \times 0.5 = 2.65$. More formally, since the intervals $(0, 1.5)$ and $(1.5, 2.0)$ are non-overlapping, the number of photon arrivals in the intervals are independent Poisson random variables.

$$\begin{aligned}\mathsf{P}(N_2 = 10 \mid N_{1.5} = 4) &= \frac{\mathsf{P}(N_{1.5} = 4, N_2 - N_{1.5} = 6)}{\mathsf{P}(N_{1.5} = 4)} \\ &= \frac{\mathsf{P}(N_{1.5} = 4)\,\mathsf{P}(N_2 - N_{1.5} = 6)}{\mathsf{P}(N_{1.5} = 4)} \\ &= \mathsf{P}(N_2 - N_{1.5} = 6) \\ &= \frac{\mathrm{e}^{-2.65}(2.65)^6}{6!} \\ &\doteq 0.034\end{aligned}$$

3. (5 points) What is the probability that the time of the second photon arrival is more than 0.2 microseconds after the arrival time of the first photon?

   Solution: Times between arrivals are independent exponential random variables. Let W_2 be the time after the first and before the second photons.

$$\mathsf{P}(W_2 > 0.2) = \int_{0.2}^{\infty} 5.3\mathrm{e}^{-5.3t}\,\mathrm{d}t = \mathrm{e}^{-(5.3)(0.2)} = \mathrm{e}^{-1.06} = 0.3465$$

An alternative solution is that the number of events in the 0.2 microseconds immediately after the first photon arrival must be zero, the expected count is $5.3 \times 0.2 = 1.06$, and the count is Poisson(1.06), so the probability is $\mathrm{e}^{-1.06}(1.06)^0/0! = \mathrm{e}^{-1.06}$.

4. (5 points) Due to an experimental error, we do not know the number of photons or any specific arrival times of photons that arrived in the time interval $(0, 2)$ microseconds. However, we did record that a total of ten photons were detected in the time interval $(1, 4)$, and that the five arrival times in time interval $(2, 4)$ were at times 2.145, 2.259, 2.994, 3.104, and 3.702 respectively. What is the probability that there were seven or fewer arrivals in the time interval $(0, 2)$?

   Solution: The answer is the probability that there are two or fewer photon arrivals in the first microsecond. The events can all be rewritten to be about the number of arrivals in the non-overlapping time intervals $(0, 1)$, $(1, 2)$, and $(2, 4)$, which are then independent. The specific times of arrivals in $(2, 4)$ are independent of the count in $(0, 2)$. Here is the derivation.

$$
\begin{aligned}
\mathsf{P}\left(N_2 \le 7 \,|\, N_4 - N_1 = 10, N_4 - N_2 = 5\right) &= \frac{\mathsf{P}\left(N_2 \le 7, N_4 - N_1 = 10, N_4 - N_2 = 5\right)}{\mathsf{P}\left(N_4 - N_1 = 10, N_4 - N_2 = 5\right)} \\
&= \frac{\mathsf{P}\left(N_1 \le 2, N_2 - N_1 = 5, N_4 - N_2 = 5\right)}{\mathsf{P}\left(N_2 - N_1 = 5, N_4 - N_2 = 5\right)} \\
&= \frac{\mathsf{P}\left(N_1 \le 2\right)\mathsf{P}\left(N_2 - N_1 = 5\right)\mathsf{P}\left(N_4 - N_2 = 5\right)}{\mathsf{P}\left(N_2 - N_1 = 5\right)\mathsf{P}\left(N_4 - N_2 = 5\right)} \\
&= \mathsf{P}\left(N_1 \le 2\right) \\
&= \mathrm{e}^{-5.3} + \mathrm{e}^{-5.3}(5.3) + \mathrm{e}^{-5.3}(5.3)^2/2 \\
&\doteq 0.1016
\end{aligned}
$$