These problems are more appropriate for homework than an exam as they are messy....
In a Bayesian model, $X_1, \ldots, X_n \mid \mu \sim \mathrm{N}(\mu, 5^2)$ with prior distribution $\mu \sim \mathrm{N}(100, 10^2)$.

1. What is $\mathsf{E}(X_1)$?

   Solution: $\mathsf{E}(X_1) = \mathsf{E}(\mathsf{E}(X_1 \mid \mu)) = \mathsf{E}(\mu) = 100$.

2. Write an expression for $\mathsf{P}(X_1 > 110)$.

   Solution: The conditional probability $\mathsf{P}(X_1 > 110 \mid \mu)$ is $1 - \Phi((110-\mu)/5)$, and the answer is to average this expression against the prior density.

$$\begin{aligned}
\mathsf{P}(X_1 > 110) &= \int_{-\infty}^{\infty} \mathsf{P}(X_1 > 110 \mid \mu)\, \pi(\mu)\, \mathrm{d}\mu \\
&= \int_{-\infty}^{\infty} \int_{110}^{\infty} \frac{1}{\sqrt{2\pi}5} \mathrm{e}^{-(x-\mu)^2/(2\cdot 25)}\, \mathrm{d}x \frac{1}{\sqrt{2\pi}10} \mathrm{e}^{-(\mu-100)^2/(2\cdot 100)}\, \mathrm{d}\mu
\end{aligned}$$

It turns out that it is possible to simplify this expression. Here is the solution to a general problem of finding $\mathsf{P}(X < t)$ where $X \mid \mu \sim \mathsf{N}(\mu, \sigma^2)$ and $\mu \sim \mathsf{N}(\mu_0, \sigma_0^2)$. The solution uses this algebraic identity found by completing the square,

$$a(\mu - c)^2 + b(\mu - d)^2 = (a+b)\left(\mu - \frac{ac+bd}{a+b}\right)^2 + \left(\frac{ab}{a+b}\right)(c-d)^2$$

and also this algebraic identity relating sums of variances and their sums of their reciprocals.

$$\sigma\sigma_0\sqrt{\sigma^{-2} + \sigma_0^{-2}} = \sqrt{\sigma^2 + \sigma_0^2}$$

This derivation leaves out a lot of the algebra, but proceeds by switching the order of integration and writing the inner integral in terms of a normal density.

$$\begin{aligned}
\mathsf{P}(X < t) &= \int_{-\infty}^{\infty} \mathsf{P}(X < t \mid \mu)\, \pi(\mu)\, \mathrm{d}\mu \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{d}x \right) \frac{1}{\sqrt{2\pi}\sigma_0} \mathrm{e}^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}}\, \mathrm{d}\mu \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma\sigma_0} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}}\, \mathrm{d}\mu \right) \mathrm{d}x \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\sigma_0^2}} \mathrm{e}^{-\frac{(x-\mu_0)^2}{2(\sigma^2+\sigma_0^2)}} \\
&\quad \times \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(\sigma^{-2}+\sigma_0^{-2})^{-1/2}} \mathrm{e}^{-\frac{\left(\mu - \frac{\sigma^{-2}x + \sigma_0^{-2}\mu_0}{\sigma^{-2}+\sigma_0^{-2}}\right)^2}{2\left(\sigma^{-2}+\sigma_0^{-2}\right)^{-1}}}\, \mathrm{d}\mu \right) \mathrm{d}x \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\sigma_0^2}} \mathrm{e}^{-\frac{(x-\mu_0)^2}{2(\sigma^2+\sigma_0^2)}}\, \mathrm{d}x \\
&= \int_{-\infty}^{(t-\mu_0)/\sqrt{\sigma^2+\sigma_0^2}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}}\, \mathrm{d}z \\
&= \Phi\left(\frac{t-\mu_0}{\sqrt{\sigma^2+\sigma_0^2}} \right)
\end{aligned}$$

Note that this derivation implies that the marginal distribution of X is $X \sim \mathsf{N}(\mu_0, \sigma^2 + \sigma_0^2)$.

It follows that the numerical answer to the original problem is then $\mathsf{P}(X_1 > 110) = 1 - \Phi((110-100)/\sqrt{100+25}) \doteq 1 - \Phi(0.8944) = 0.1855$.

3. Suppose that for sample s, $n = 50$ and $\bar{x} = 92$. Find the posterior density of μ.

Solution:

$$\begin{aligned}
\pi(\mu \,|\, s) &\propto \pi(\mu) f_\mu(s) \\
&\propto \mathrm{e}^{-(\mu-100)^2/(2\cdot 10^2)} \mathrm{e}^{-(50/2\cdot 5^2)(92-\mu)^2} \\
&\propto \mathrm{e}^{-(\mu-\mu_1)^2/(2\eta)}
\end{aligned}$$

where

$$\mu_1 = \left(\frac{0.01}{2.01}\right) 100 + \left(\frac{2}{0.01}\right) 92 = 92.0398$$

and

$$\eta = \left(\frac{1}{100} + \frac{50}{25}\right)^{-1} = 0.4975$$

which implies that $\mu \,|\, s \sim \mathsf{N}(\mu_1, \eta)$.

4. Write an expression for $\mathsf{P}(X_{51} > 110 \,|\, s)$.

Solution: Here we integrate the probability against the posterior density instead of the prior density.

$$\mathsf{P}(X_{51} > 110 \,|\, s) = \int_{-\infty}^{\infty} \mathsf{P}(X_{51} > 100 \,|\, \mu)\, \pi(\mu \,|\, s)\, \mathrm{d}\mu$$

By the work from before, $X_{51} \,|\, s \sim \mathsf{N}(\mu_1, \sigma^2 + \eta)$ which for this data implies that $X_{51} \,|\, s \sim \mathsf{N}(92.0398, 25.4975)$ so the probability is $1 - \Phi((110 - 92.0398)/\sqrt{25.4975}) = 1 - \Phi(3.557) \doteq 0.00019$.