16.3.17 Using \( c(x_1, \ldots, x_n) = \bar{x} + k(\sigma_0/\sqrt{n}) \), we have that \( k \) satisfies
\[
P(\mu \leq \bar{x} + k(\sigma_0/\sqrt{n})) = P\left(\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}} \geq -k\right) = P\left(X \geq -k\right) \geq \gamma
\]
So \( k = -z_{1-\gamma} = z_\gamma \), i.e. the \( \gamma \)-percentile of a \( N(0,1) \) distribution.

6.3.18 \( H_0 : \mu \leq \mu_0 \)
\[
P - value = P_\mu(\bar{X} - \mu \geq \bar{x} - \mu_0) = P_\mu\left(\frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} \geq \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}\right) = 1 - P_\mu\left(\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} \right)
\]

2. (a) \( F_\theta(x) = \int_0^x \theta(1 + t)^{-\theta} dt = 1 - (1 + x)^{-\theta} \)
(b) \( \int_0^\infty \theta(1 + x)^{-\theta} dx = -(1 + x)^{-\theta} \bigg|_{x=0}^{x=\infty} = 1 \)
(c) \( f(\theta \mid x_1, \ldots, x_n) = \theta^n \prod_{i=1}^n (1 + x_i)^{-(\theta+1)} \)
\( l(\theta \mid s) = n \log(\theta) - (\theta + 1) \sum_{i=1}^n \log(1 + x_i) \)
\[ \Rightarrow \hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^n \log(1+x_i)} \]
The numerical value of \( \hat{\theta}_{MLE} \) is 0.768
(d) One possible minimal sufficient statistic is \( \Pi_{i=1}^n (1 + x_i) \)
(e) \( l(\theta \mid s) = n \log(\theta) - (\theta + 1) \sum_{i=1}^n \log(1 + x_i) \)
\[ \Rightarrow \frac{-d^2 l(\theta|s)}{d\theta^2} = \frac{n}{\theta^2} \]
Observed Fisher Information: \[ -\frac{d^2 l(\theta|s)}{d\theta^2} \bigg|_{\theta=\hat{\theta}} = \frac{n}{\hat{\theta}^2} = 15/0.768^2 = 25.43 \]
Plug in Fisher Information: \( E\left(-\frac{d^2 l(\theta|s)}{d\theta^2}\right) \bigg|_{\theta=\hat{\theta}} = 25.43 \)
(f) \( \hat{\theta} \overset{D}{\rightarrow} N(\theta, 1/nI(\theta)) \), where \( nI(\theta) = \frac{n}{\theta^2} \)
(g) Replace \( \theta \) by \( \hat{\theta}_{MLE} \) found in part (c) in the density function to get \( f_\hat{\theta}(x) \), then simulate a large number of samples from \( f_\hat{\theta}(x) \). Compute the numerical \( \hat{\theta} \) for each sample and calculate the mean of these \( \hat{\theta} \), then estimate the bias of \( \hat{\theta} \) by taking the difference of \( \hat{\theta}_{MLE} \) and the mean.

36.5.2 \( ln f_\theta(x) = \log \theta - \theta x \)
\[ \frac{d^2 ln f_\theta(x)}{d\theta^2} = -\frac{1}{\theta^2} \text{, since } \theta > 0, \text{ thus } -\frac{1}{\theta^2} \text{ always exists.} \]

(6.5.3)
\[
\int_0^\infty \frac{dln f_\theta(x)}{d\theta} f_\theta(x) dx = \int_0^\infty \left(\frac{1}{\theta} - x\right) e^{-\theta x} dx
\]
\[ = \int_0^\infty e^{-\theta x} dx - e^{-\theta x} \theta x e^{-\theta x} dx
\]
\[ = \frac{1}{\theta} - \frac{\Gamma(2)}{\theta} \int_0^\infty \frac{\theta^2}{\Gamma(2)} x e^{-\theta x} dx
\]
\[ = \frac{1}{\theta} - \frac{1}{\theta} = 0 \]
\[ \int_0^\infty \frac{d}{d\theta} f_\theta(x) \frac{d}{d\theta} f_\theta(x) \, dx = \int_0^\infty \frac{d}{d\theta} \frac{1}{\theta x} - x \theta e^{-\theta x} \, dx \]
\[ = -2 \int_0^\infty x e^{-\theta x} \, dx + \theta \int_0^\infty x^2 e^{-\theta x} \, dx \]
\[ = -\frac{2\Gamma(2)}{\theta^2} \int_0^\infty \frac{x^2}{\Gamma(2)} \theta e^{-\theta x} \, dx + \frac{\Gamma(3)}{\theta^2} \int_0^\infty \frac{\theta^3}{\Gamma(3)} x^2 e^{-\theta x} \, dx \]
\[ = -\frac{2}{\theta^2} + \frac{2}{\theta^2} = 0 \]

4. \( \pi(\theta \mid s) = \frac{\pi(\theta) f_\theta(s)}{m(s)} \), for this problem, both \( \theta \) and \( s \) are discrete, thus \( m(s) = \sum_{\theta=1}^{3} pi(\theta) f_\theta(s) \)

Let’s find \( m(s) \) first:

If \( s = 1 \), \( m(s) = \frac{1}{5} \times \frac{1}{12} + \frac{2}{5} \times \frac{1}{3} + \frac{2}{5} \times \frac{1}{4} = \frac{8}{15} \)

If \( s = 2 \), \( m(s) = \frac{1}{5} \times \frac{1}{2} + \frac{2}{5} \times \frac{2}{3} + \frac{2}{5} \times \frac{1}{4} = \frac{13}{15} \)

Now, let’s find \( \pi(\theta \mid s) \):

When \( s = 1 \)

If \( \theta = 1 \), then \( \pi(\theta \mid s = 1) = \frac{\pi(\theta = 1) f_1(s = 1)}{m(s = 1)} = \frac{3}{16} \)

If \( \theta = 2 \), then \( \pi(\theta \mid s = 1) = \frac{\pi(\theta = 2) f_2(s = 1)}{m(s = 1)} = \frac{1}{4} \)

If \( \theta = 3 \), then \( \pi(\theta \mid s = 1) = \frac{\pi(\theta = 3) f_3(s = 1)}{m(s = 1)} = \frac{9}{16} \)

When \( s = 2 \)

If \( \theta = 1 \), then \( \pi(\theta \mid s = 2) = \frac{\pi(\theta = 1) f_1(s = 2)}{m(s = 2)} = \frac{3}{14} \)

If \( \theta = 2 \), then \( \pi(\theta \mid s = 2) = \frac{\pi(\theta = 2) f_2(s = 2)}{m(s = 2)} = \frac{4}{7} \)

If \( \theta = 3 \), then \( \pi(\theta \mid s = 2) = \frac{\pi(\theta = 3) f_3(s = 2)}{m(s = 2)} = \frac{3}{14} \)

5. \( \pi(\theta) = \frac{\lambda^\alpha e^{-\lambda \theta}}{\Gamma(\alpha)} e^{-\theta} \), \( f_\theta(s) = \frac{\theta^{\Sigma x_i}}{\prod_{x_i=1}^{\text{Pi}} e^{-n \theta}} \)

\( \pi(\theta \mid x) \propto \pi(\theta) f_\theta(x) \implies \pi(\theta \mid x) \propto \frac{\lambda^{\alpha+n} e^{\alpha \Sigma x_i + \Sigma x_i - 1}}{\Gamma(\alpha + \Sigma x_i)} e^{-(\lambda+n) \theta} \)

Therefore, the posterior distribution of \( \theta \) is Gamma(\( \alpha + \Sigma x_i, \lambda + n \))

6. Given the sample data, the posterior distribution of \( \theta \) is Gamma(109, 8.2)

The 95% credible region for \( \theta \) is (10.915, 15.902):

\> qgamma(c(0.025, 0.975), 109, 8.2)
\> [1] 10.91468 15.90155

The prior and posterior densities of \( \theta \) are given below

\> ggamma(c(0,40),2,0.2)
\> ggamma(c(0,40),109,8.2)