STAT310 - HWK Solution 5

1. 6.3.17 Using $c(x_1, ..., x_n) = \bar{x} + k(\sigma_0/\sqrt{n})$, we have that k satisfies

$$P(\mu \leqslant \bar{x} + k(\sigma_0/\sqrt{n})) = P(\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}} \geqslant -k) = P(X \geqslant -k) \geqslant \gamma$$

So $k = -z_{1-\gamma} = z_\gamma$, i.e. the γ-percentile of a $N(0, 1)$ distribution.

6.3.18 $H_0 : \mu \leqslant \mu_0$

$$P - value = P_\mu(\bar{X} - \mu \geqslant \bar{x} - \mu_0) = P_\mu(\frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} \geqslant \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}) = P_\mu(Z \geqslant \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}})$$

$$= 1 - P_\mu(Z \leqslant \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}) = 1 - \Phi(\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}})$$

2. (a) $F_\theta(x) = \int_0^x \theta(1 + t)^{-\theta-1} dt = 1 - (1 + x)^{-\theta}$

(b) $\int_0^\infty \theta(1 + x)^{-\theta-1} dx = -(1 + x)^{-\theta} \mid_0^\infty = 1$

(c) $f(\theta \mid x_1, ..., x_n) = \theta_n[\Pi_{i=1}^n (1 + x_i)]^{-(\theta+1)}$
$l(\theta \mid s) = nlog(\theta) - (\theta + 1)\Sigma_{i=1}^n log(1 + x_i)$
$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{\Sigma_{i=1}^n log(1+x_i)}$
The numerical value of $\hat{\theta}_{MLE}$ is 0.768

(d) One possible minimal sufficient statistic is $\Pi_{i=1}^n (1 + x_i)$

(e) $l(\theta \mid s) = nlog(\theta) - (\theta + 1)\Sigma_{i=1}^n log(1 + x_i)$
$-\frac{d^2 l(\theta|s)}{d\theta^2} = \frac{n}{\theta^2}$
Observed Fisher Information: $-\frac{d^2 l(\theta|s)}{d\theta^2} \mid_{\theta=\hat{\theta}} = \frac{n}{\hat{\theta}^2} = 15/0.768^2 = 25.43$
Plug in Fisher Information: $E(-\frac{d^2 l(\theta|s)}{d\theta^2}) \mid_{\theta=\hat{\theta}} = 25.43$

(f) $\hat{\theta} \xrightarrow{D} N(\theta, \frac{1}{nI(\hat{\theta})})$, where $nI(\hat{\theta}) = \frac{n}{\hat{\theta}^2}$

(g) Replace θ by $\hat{\theta}_{MLE}$ found in part (c) in the density function to get $f_{\hat{\theta}}(x)$, then simulate a large number of samples from $f_{\hat{\theta}}(x)$. Compute the numerical $\hat{\theta}$ for each sample and calculate the mean of these $\hat{\theta}$, then estimate the bias of $\hat{\theta}$ by taking the difference of $\hat{\theta}_{MLE}$ and the mean.

3. (6.5.2) $lnf_\theta(x) = log\theta - \theta x$
$\frac{d^2 lnf_\theta(x)}{d\theta^2} = -\frac{1}{\theta^2}$, since $\theta > 0$, thus $-\frac{1}{\theta^2}$ always exists.

(6.5.3)

$$\int_0^\infty \frac{dlnf_\theta(x)}{d\theta} f_\theta(x)dx = \int_0^\infty (\frac{1}{\theta} - x)\theta e^{-\theta x} dx$$

$$= \int_0^\infty e^{-\theta x} dx - e^{-\theta x}\theta x e^{-\theta x} dx$$

$$= \frac{1}{\theta} - \frac{\Gamma(2)}{\theta} \int_0 \infty \frac{\theta^2}{\Gamma(2)} x e^{-\theta x} dx$$

$$= \frac{1}{\theta} - \frac{1}{\theta} = 0$$

(6.5.4)

$$\int_0^\infty \frac{d}{d\theta}\frac{dln f_\theta(x)}{d\theta} f_\theta(x)dx = \int_0^\infty \frac{d}{d\theta}\frac{1}{\theta} - x)\theta e^{-\theta x}dx$$
$$= -2\int_0^\infty xe^{-\theta x}dx + \theta\int_0^\infty x^2 e^{-\theta x}dx$$
$$= -\frac{2\Gamma(2)}{\theta^2}\int_0^\infty \frac{\theta^2}{\Gamma(2)}xe^{-\theta x}dx + \frac{\Gamma(3)}{\theta^2}\int_0^\infty \frac{\theta^3}{\Gamma(3)}x^2 e^{-\theta x}dx$$
$$= -\frac{2}{\theta^2} + \frac{2}{\theta^2} = 0$$

4. $\pi(\theta \mid s) = \frac{\pi(\theta)f_\theta(s)}{m(s)}$, for this problem, both θ and s are discrete, thus $m(s) = \Sigma_{\theta=1}^3 pi(\theta)f_\theta(s)$

   Let's find $m(s)$ first:
   If $s = 1$, $m(s) = \frac{1}{5} * \frac{1}{2} + \frac{2}{5} * \frac{1}{3} + \frac{2}{5} * \frac{3}{4} = \frac{8}{15}$
   If $s = 2$, $m(s) = \frac{1}{5} * \frac{1}{2} + \frac{2}{5} * \frac{2}{3} + \frac{2}{5} * \frac{1}{4} = \frac{7}{15}$
   Now, let's find $\pi(\theta \mid s)$:
   When s=1
   If $\theta = 1$, then $\pi(\theta \mid s = 1) = \frac{\pi(\theta=1)f_1(s=1)}{m(s=1)} = \frac{3}{16}$
   If $\theta = 2$, then $\pi(\theta \mid s = 1) = \frac{\pi(\theta=2)f_2(s=1)}{m(s=1)} = \frac{1}{4}$
   If $\theta = 3$, then $\pi(\theta \mid s = 1) = \frac{\pi(\theta=3)f_3 S(s=1)}{m(s=1)} = \frac{9}{16}$

   When s=2
   If $\theta = 1$, then $\pi(\theta \mid s = 2) = \frac{\pi(\theta=1)f_1(s=2)}{m(s=2)} = \frac{3}{14}$
   If $\theta = 2$, then $\pi(\theta \mid s = 2) = \frac{\pi(\theta=2)f_2(s=2)}{m(s=2)} = \frac{4}{7}$
   If $\theta = 3$, then $\pi(\theta \mid s = 2) = \frac{\pi(\theta=3)f_3(s=2)}{m(s=2)} = \frac{3}{14}$

5. $\pi(\theta) = \frac{\lambda^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda\theta}$, $f_\theta(s) = \frac{\theta^{\Sigma x_i}}{\Pi_{i=1}^n x_i}e^{-n\theta}$
   $\pi(\theta \mid x) \propto \pi(\theta)f_\theta(x) \Longrightarrow \pi(\theta \mid x) \propto \frac{(\lambda+n)^{\alpha+\Sigma x_i}\theta^{\alpha+\Sigma x_i - 1}}{\Gamma(\alpha+\Sigma x_i)}e^{-(\lambda+n)\theta}$
   Therefore, the posterior disctribution of θ is $Gamma(\alpha + \Sigma x_i, \lambda + n)$

6. Given the sample data, the posterior distribution of θ is $Gamma(109, 8.2)$
   The 95% credible region for θ is (10.915, 15.902):

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> qgamma(c(0.025,0.975), 109, 8.2)
[1] 10.91468 15.90155
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   The prior and posterior densities of θ are given below

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> ggamma(c(0,40),2,0.2)
> ggamma(c(0,40),109,8.2)
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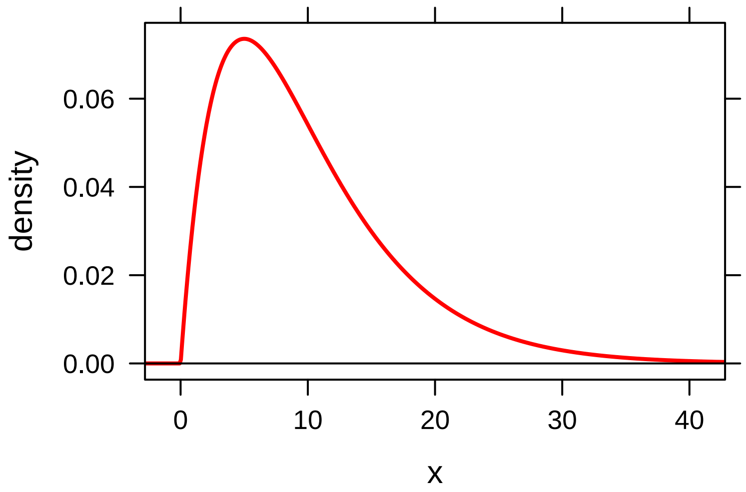
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0.02
0.04
0.06
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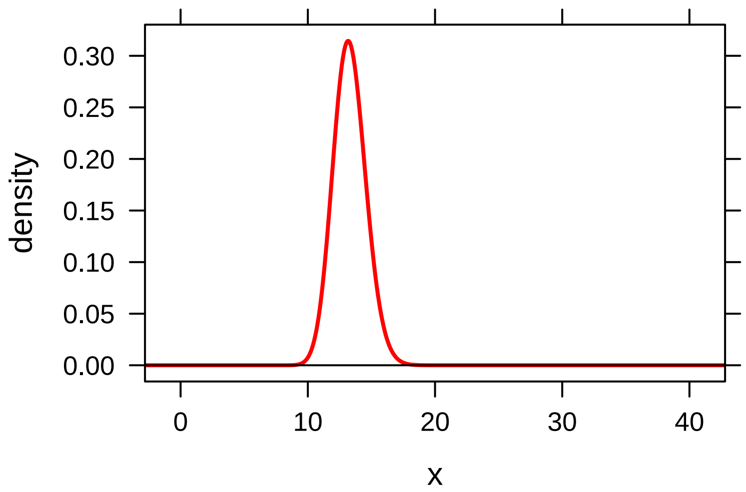
10
20
30
40
x


density
0.00
0.05
0.10
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0.25
0.30
0
10
20
30
40
x