STAT310 - HWK Solution 6

1.
```
> source("stat310.R")
> library(boot)
> tornado = read.table("wi-tornado.txt", header=T)
> with(tornado, boot(count, get.q75,R=10000))

ORDINARY NONPARAMETRIC BOOTSTRAP

Call: boot(data = count, statistic = get.q75, R = 10000)

Bootstrap Statistics :
    original  bias    std. error
t1*    32.75 -1.1239     4.08338

> 32.75+1.96*4.0833
[1] 40.75327
>  32.75-1.96*4.0833
[1] 24.74673
```
The 95% confidence interval is [24.75,40.75]

2. Confidence intervals

   (a) let $W = \Sigma X_i$, where $X_i \sim Exponential(\theta) = Gamma(1, \theta)$, recall that $W \sim Gamma(n, \theta)$
   $Y = \theta\bar{X} = \frac{\theta}{n}W$, thus $Z = \frac{nY}{\theta}$
   $f_Y(y) = f_Z(\frac{ny}{\theta})|\frac{dz}{dy}| = \frac{\theta^n}{\Gamma(n)}(\frac{ny}{\theta})^{n-1}E^{-\theta(\frac{ny}{\theta})}\frac{n}{\theta} = \frac{n^n}{\Gamma(n)}y^{n-1}e^{-ny}$. Therefore, Y$\sim Gamma(n, n)$
   Let a be the 0.025 quantile and b be the 0.975 quantile of $Gamma(n, n)$ distribution. Then $P(a < \theta\bar{X} < b) = 0.95$, the 95% confidence interval for θ is $[a/\bar{X}, b/\bar{X}] = [0.1454, 0.3532]$

   (b) $\hat{\theta} \pm z\sqrt{\frac{\hat{\theta}^2(n+2)}{(n-1)(n-2)}} = \frac{1}{4.2} \pm 1.96\sqrt{0.003646683} = [0.1197, 0.3565]$

   (c) $f_\theta(x) = \theta e^{-\theta x} \Longrightarrow -\frac{d^2 log f_\theta(x)}{d\theta^2} = \frac{1}{\theta^2}$, thus $I(\theta) = E(-\frac{d^2 log f_\theta(x)}{d\theta^2}) = \frac{1}{\theta^2}$
   $\hat{\theta} \pm z/\sqrt{nI(\hat{\theta})} = \frac{1}{4.2} \pm 1.96/\sqrt{352.8} = [0.1337, 0.3424]$

   (d) $MSE(\bar{\theta}) = var(\bar{\theta}) + bias(\bar{\theta})^2$. Let $T = \Sigma X_i, T \sim Gamma(n, \theta)$
   Since $\bar{\theta}$ is an unbiased estimator of θ, $bias(\bar{\theta}) = 0, MSE(\bar{\theta}) = var(\bar{\theta})$

$$var(\bar{\theta}) = E(\bar{\theta}^2) - E(\bar{\theta})^2 = \frac{(n-1)^2}{n^2}E(\frac{1}{\bar{X}^2}) - \theta^2$$
$$= (n-1)^2 E(\frac{1}{T^2}) = \frac{(n-1)^2}{n^2}\int_0^\infty \frac{\theta^n}{\gamma(n)}t^{n-3}e^{-\theta t}dt - \theta^2$$
$$= \frac{\theta^2}{n-2}$$

   Thus, $MSE(\bar{\theta}) = \frac{\theta^2}{n-2}$, $\bar{\theta} \pm z\sqrt{MSE(\bar{\theta})} = 0.2662 \pm 1.96\sqrt{0.2662/18} = [0.1217, 0.3307]$

(e) Range of exact method: 0.2708
Range of MSE method: 0.2368
Range of Fisher Information method: 0.2087
Range of Unbiased Estimator method: 0.2090
The shortest range is 0.2078 with the exact method.

3.
```
> source("exponential.R")
> exponential.sim(10000,5)
$n [1] 5

$contain
     Exact    MLE/MSE MLE/Fisher   Unbiased
       945        996        954        937

$range
     Exact    MLE/MSE MLE/Fisher   Unbiased
  2.213551   3.844895   2.251340   2.325174
```
The exact and fisher methods have true coverage probability close to 0.95.

4.
```
> exponential.sim(10000,100)
[1] 100

$contain
     Exact    MLE/MSE MLE/Fisher   Unbiased
       955        951        948        947

$range
     Exact    MLE/MSE MLE/Fisher   Unbiased
 0.3945387  0.4048923  0.3948845  0.3949047
```
As sample size increases, all methods have true coverage probabilities close to 0.95.