Assignment #1 — Solution

1. (a)

Solution: The events $\{Y = 0\} = \{W_1 > 1\}$ since the time of the first event is after time 1 if and only if there are no events in the interval $(0, 1]$. From the exponential point of view, the probability is

$$\mathsf{P}(W_1 > 1) = \int_1^\infty 2e^{-2x}\,dx = e^{-2\cdot 1} \doteq 0.1353 .$$

From the Poisson point of view, $Y \sim$ Poisson(2).

$$\mathsf{P}(Y = 0) = \frac{e^{-2}2^0}{0!} \doteq 0.1353 .$$

Using R,

```
> c(1 - pexp(1, 2), dpois(0, 2))
[1] 0.1353353 0.1353353
```

(b)

Solution: The events $\{Y = 1\}$ and $\{W_1 < 1 \cap W_1 + W_2 > 1\}$ since the time of the first event must be before time 1 and the time of the second event must be after. The probability from the exponential point of view is

$$\begin{aligned}\mathsf{P}(W_1 < 1 \cap W_1 + W_2 > 1) &= \int_0^1 \int_{1-x}^\infty 4e^{-2x}e^{-2y}\,dy\,dx \\ &= \int_0^1 4e^{-2x}\left(e^{-2(1-x)}/2\right)dx \\ &= \int_0^1 2e^{-2}\,dx \\ &= 2e^{-2} \\ &\doteq 0.2707 .\end{aligned}$$

The Poisson probability is

$$\mathsf{P}(Y = 1) = \frac{e^{-2}2^1}{1!} \doteq 0.2707 .$$

(c)

Solution: Sorry for the incomplete example! Use `table()` to summarize the results.

```
> table(simPP(100000))/100000
      0       1       2       3       4       5       6       7       8       9      10
0.13719 0.26812 0.26985 0.17891 0.09209 0.03665 0.01253 0.00361 0.00083 0.00017 0.00004
     11
0.00001
```

(d)

```
> round(dpois(0:8, 2), 5)
[1] 0.13534 0.27067 0.27067 0.18045 0.09022 0.03609 0.01203 0.00344 0.00086
```

The exact Poisson probabilities and the estimates from sample frequencies are very close.

2.

Solution:

(a) X_1 and X_3 are independent because they count events in disjoint intervals. X_2 is not independent of either X_1 or X_3 as its interval intersects with both.

(b) $\mathsf{E}(X_i) = \lambda t = 2 \times 2 = 4$ for $i = 1, 2, 3$.

(c) $\mathsf{P}(X_1 = 2) = \frac{e^{-4}4^2}{2!} \doteq 0.1465$

(d) Let $Y_1 = N_1$, $Y_2 = N_2 - N_1$, and $Y_3 = N_3 - N_2$. Then these $\{Y_i\}$ are independent Poisson(2) random variables and $X_1 = Y_1 + Y_2$ while $X_2 = Y_2 + Y_3$.

$$\begin{aligned}
\mathsf{P}(X_1 = 2 \,|\, X_2 = 1) &= \frac{\mathsf{P}(X_1 = 2 \cap X_2 = 1)}{\mathsf{P}(X_2 = 1)} \\
&= \frac{\mathsf{P}(Y_1 + Y_2 = 2 \cap Y_2 + Y_3 = 1)}{\mathsf{P}(X_2 = 1)} \\
&= \frac{\mathsf{P}(Y_1 = 1, Y_2 = 1, Y_3 = 0) + \mathsf{P}(Y_1 = 2, Y_2 = 0, Y_3 = 1)}{\mathsf{P}(X_2 = 1)} \\
&= \frac{(2e^{-2})^2(e^{-2}) + (2^2e^{-2}/2)(e^{-2})(2e^{-2})}{4e^{-4}} \\
&= 2e^{-2} \\
&= 0.2707
\end{aligned}$$

In R, this is

```
> (prod(dpois(c(1, 1, 0), 2)) + prod(dpois(c(2, 0, 1), 2)))/dpois(1, 4)
[1] 0.2706706
```

(e) $\mathsf{P}(X_1 = 2 \,|\, X_3 = 1) = \mathsf{P}(X_1 = 2) = e^{-4}4^2/2 = 8e^{-4} \doteq 0.1465$.

3.

Solution: Note that as long as $0 < \theta < 1$ that $t = -\log(1-\theta)$. By the method of maximum likelihood, $L(\theta) = \binom{10}{3}\theta^3(1-\theta)^7$ and $\ell(\theta) = \log L(\theta) = \log\binom{10}{3} + 3\log\theta + 7\log(1-\theta)$. Setting the derivative equal to and solving yields the estimate.

$$\begin{aligned}
\ell'(\theta) &= \frac{3}{\theta} - \frac{7}{1-\theta} = 0 \\
3 - 3\theta &= 7\theta \\
\hat{\theta} &= \frac{3}{10}
\end{aligned}$$

Hence, $\hat{t} = -\log(7/10) = 0.3567$.

An alternative solution (using the method of moments) is $\mu = 10\theta \approx 3 = X$. Thus $\theta \approx 3/10$ which agrees with the maximum likelihood approach.

4.

Solution: By the method of maximum likelihood,

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{5} \binom{10}{x_i} \theta^{x_i} (1-\theta)^{10-x_i} \\
&= \left(\prod_{i=1}^{5} \binom{10}{x_i} \right) \theta^{\sum_i x_i} (1-\theta)^{50-\sum_i x_i} \\
\ell(\theta) &= \log C + 13 \log \theta + 37 \log(1-\theta) \\
\ell'(\theta) &= 0 + \frac{13}{\theta} - \frac{37}{1-\theta} = 0 \\
\hat{\theta} &= \frac{13}{50}
\end{aligned}$$

So that $\hat{t} = -\log(13/50) = 1.3471$.

An alternative solution (using the method of moments) is $\mu = 10\theta \approx 13/5 = \overline{X}$. Thus $\theta \approx 13/50$ which agrees with the maximum likelihood approach.