1. Discrete random variables $X_1, X_2, X_3, \ldots$ are independent and identically distributed with these probabilities: $\mathsf{P}(X_i = 0) = 0.6$, $\mathsf{P}(X_i = 5) = 0.3$, and $\mathsf{P}(X_i = 15) = 0.1$.

   (a) (8 points) Find $\mathsf{E}(X_i)$ and $\mathsf{Var}(X_i)$.
   Solution: The expected value is

$$\mathsf{E}(X_i) = 0(0.6) + 5(0.3) + 15(0.1) = 3.$$

   The variance is $\mathsf{Var}(X_i) = \mathsf{E}(X_i^2) - 3^2$.

$$\mathsf{E}(X_i^2) = 0(0.6) + 25(0.3) + 225(0.1) = 30$$

   so

$$\mathsf{Var}(X_i) = 30 - 9 = 21$$

   (b) (8 points) Let $S_n = \sum_{i=1}^{n} X_i$. Approximate $\mathsf{P}(1470 \le S_{500} \le 1550)$.
   Solution: By the central limit theorem, $S_{500} \approx \mathsf{N}(1500, (\sqrt{10,500})^2)$. We can improve the approximation a little by noting that S_n is a multiple of 5, so that $\mathsf{P}(S_n = 1470)$ is approximated by the area between 1467.5 and 1472.5, for example. Also, $\mathrm{SD}(S_{500}) = 10\sqrt{105} \doteq 102.4695$.

$$\begin{aligned}
\mathsf{P}(1470 \le S_{500} \le 1550) &= \mathsf{P}(1467.5 < S_{500} < 1552.5) \\
&= \mathsf{P}\left(\frac{1467.5 - 1500}{10\sqrt{105}} < \frac{S_{500} - 1500}{10\sqrt{105}} < \frac{1552.5 - 1500}{10\sqrt{105}}\right) \\
&\approx \mathsf{P}(-0.32 < Z < 0.51) \\
&= 0.695 - 0.3745 = 0.3205
\end{aligned}$$

   (c) (4 points) Find a number c so that $\mathsf{P}(S_{500} < c) \approx 0.84$.
   Solution: If

$$\mathsf{P}(S_{500} < c) \approx 0.84$$

   then

$$\mathsf{P}\left(Z < \frac{c - 1500}{10\sqrt{105}}\right) \approx 0.84$$

   and it follows that

$$\frac{c - 1500}{10\sqrt{105}} \approx 0.99$$

   and so

$$c = 1500 + 0.99(10\sqrt{105}) \doteq 1601$$

   Since S_{500} is exactly a multiple of 5, we may conclude that the probability that S_{500} is 1600 or below is close to 0.84.

2. The moment generating function of the Binomial(n, θ) distribution is $m(t) = \left((1-\theta) + \theta \mathrm{e}^t\right)^n$. Let $Y_n \sim$ Binomial$(n, 1/n)$ and $Y \sim$ Poisson(1), so that $\mathsf{P}(Y = k) = \mathrm{e}^{-1}/k!$ for $k = 0, 1, 2, 3, \ldots$.

(a) (3 points) Find the moment generating function of Y.

Solution:

$$m_Y(t) = \sum_{k=0}^{\infty} \mathrm{e}^{tk}\frac{\mathrm{e}^{-1}}{k!} = \mathrm{e}^{-1}\sum_{k=0}^{\infty}\left(\frac{\mathrm{e}^t}{k!}\right) = \mathrm{e}^{-1}\mathrm{e}^{\mathrm{e}^t} = \mathrm{e}^{\mathrm{e}^t-1}$$

(b) (2 points) Show that $Y_n \xrightarrow{D} Y$ by showing that the moment generating functions converge. *Hint: Recall that* $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \mathrm{e}^x$ *for any* x.

Solution: Let $m_n(t) = \mathsf{E}(\mathrm{e}^{tY_n}) = \left((1-\frac{1}{n}) + \frac{1}{n}\mathrm{e}^t\right)^n$. Then,

$$\begin{aligned}
\lim_{n\to\infty} m_n(t) &= \lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)+\frac{1}{n}\mathrm{e}^t\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{\mathrm{e}^t-1}{n}\right)^n \\
&= \mathrm{e}^{\mathrm{e}^t-1}
\end{aligned}$$

Since this is the moment generating function of Y, it follows that $Y_n \xrightarrow{D} Y$.