1. Random variables $X$ and $Y$ have joint density $f_{X,Y}(x,y) = cxy$ for $x > 0, y < 2, 2x < y$ and 0 otherwise.

(a) (8 points) Find $c$.

Solution: Note that $0 < x < 1$ and for fixed $x$, $2x < y < 2$.

\[
1 = \int_0^1 \int_{2x}^2 cxy \, dy \, dx
\]
\[
= c \int_0^1 \frac{x}{2} \left( \frac{y^2}{2} \right)_{y=2x} \, dx
\]
\[
= c \int_0^1 \frac{x}{2} \left( y^2 - 2x^2 \right)_{y=2x} \, dx
\]
\[
= c \left. \left( x^3 - 2x^4 \right) \right|_{x=0}
\]
\[
= c / 2
\]

Thus, $c = 2$.

(b) (8 points) Find the marginal density of $X$.

Solution: Integrate out $y$ from the joint density.

\[
f_X(x) = \int_{2x}^2 2xy \, dy = x \left( \frac{y^2}{2} \right)_{y=2x} = 4x - 4x^3
\]

The density is $f_X(x) = 4x - 4x^3$, $0 < x < 1$.

(c) (8 points) Let $Z = 2X/Y$. Find the density of $Z$.

Solution: Here I will use the bivariate change of variable method. Let $W = X$ also. (Other choices for $W$ are legitimate.) Note that the inverse relationship is

\[
X = W, \quad Y = \frac{2W}{Z}
\]

The Jacobian derivative is

\[
\frac{\partial(x,y)}{\partial(w,z)} = \det \begin{pmatrix} 1 & 2/z \\ 0 & -2w/z^2 \end{pmatrix} = \frac{-2w}{z^2}
\]

so the joint density of $W$ and $Z$ is

\[
f_{W,Z}(w,z) = \frac{-2w}{z^2} \left( 2w(2w/z) \right) = 8w^3/z^3, \quad w > 0, z < 1, w < z
\]

The domain of the density is found by solving

\[
w > 0, \quad 2w/z < 2, \quad \text{and} \quad 2w < 2w/z
\]

This implies $w > 0, w < z,$ and $z < 1$. These inequalities imply that the marginal density of $Z$ is $0 < z < 1$ since $z < 1$ and $z > w > 0$. We then get the density of $Z$ by integrating out $W$.

\[
f_Z(z) = \int_0^z 8w^3/z^3 \, dw
\]
\[
= \left( \frac{2w^4}{z^3} \right)_{w=0}^{z}
\]
\[
= 2z, \quad 0 < z < 1
\]
So, $f_Z(z) = 2z, \ 0 < z < 1$

(d) (1 point) $Z$ has one of the named distributions we have seen in class. Identify this distribution and the associated parameter values.

Solution: $Z$ has an absolutely continuous distribution between 0 and 1 of the form $Cz^{\alpha-1}(1-z)^{\beta-1}$ so $Z \sim \text{Beta}(2, 1)$. 