1. A bucket contains 3 red and 2 blue balls. A ball is drawn uniformly at random and replaced in the bucket. This is repeated until each color appears at least once. Let $X$ be the number of red and $Y$ be the number of blue drawn balls. Let $Z = X + Y$ be the total number of drawn balls. Recall that infinite geometric series can be simplified: 
$$
\sum_{k=a}^{\infty} r^k = \frac{r^a}{1 - r} \text{ when } |r| < 1.
$$

Solution: Note that each possible sequence of balls is either one or more red balls followed by a single blue ball or one or more blue ball followed by a single red ball. If the first ball is red, then the additional number of red balls is distributed Geometric$(2/5)$. If the first ball is blue, then the additional number of blue balls is distributed Geometric$(3/5)$.

(a) (3 points) Find $P(X = 1 \cap Y = 1)$.

Solution: Either red/blue or blue/red.

$$
P(X = 1 \cap Y = 1) = \left( \frac{3}{5} \right) \left( \frac{2}{5} \right) + \left( \frac{2}{5} \right) \left( \frac{3}{5} \right) = \frac{12}{25}
$$

(b) (3 points) Find $P(Z = 3)$.

Solution: Either red/red/blue or blue/blue/red.

$$
P(Z = 3) = \left( \frac{3}{5} \right) \left( \frac{3}{5} \right) \left( \frac{2}{5} \right) + \left( \frac{2}{5} \right) \left( \frac{2}{5} \right) \left( \frac{3}{5} \right) = \frac{30}{125} = \frac{6}{25}
$$

(c) (5 points) Find the joint probability $P(X = x, Y = y)$ for all $x, y$.

Solution: 
$$
P(X = x, Y = y) = \begin{cases} 
\frac{12}{25} & \text{if } x = 1, y = 1 \\
\left( \frac{2}{5} \right)^y \left( \frac{3}{5} \right) & \text{if } x = 1, \ y = 2, 3, 4, \ldots \\
\left( \frac{3}{5} \right)^x \left( \frac{2}{5} \right) & \text{if } x = 2, 3, 4, \ldots, \ y = 1 \\
0 & \text{otherwise}
\end{cases}
$$

(d) (5 points) Find $P(X \text{ is even} | Y = 1)$.

Solution: Use the definition of conditional probability.

$$
P(X \text{ is even} | Y = 1) = \frac{P(X \text{ is even} \cap Y = 1)}{P(Y = 1)}
$$

If there is exactly one blue ball, either the sequence is blue/red or the first ball is red. Thus,

$$
P(Y = 1) = \left( \frac{2}{5} \right) \left( \frac{3}{5} \right) + \frac{3}{5} = \frac{21}{25}
$$

Alternatively,

$$
P(Y = 1) = \sum_{x=1}^{\infty} P(X = x, Y = 1) = \frac{12}{25} + \sum_{x=2}^{\infty} \left( \frac{3}{5} \right)^x \left( \frac{2}{5} \right) = \frac{12}{25} + \left( \frac{2}{5} \right) \left( \frac{9}{25} \frac{1 - 3/5}{1 - 9/25} \right) = \frac{21}{25}.
$$

When $X$ is even, $Y = 1$, so $P(X \text{ is even} \cap Y = 1) = P(X \text{ is even})$. Find this probability by summing.

$$
P(X \text{ is even}) = \sum_{k=1}^{\infty} P(X = 2k, Y = 1) = \sum_{k=1}^{\infty} \left( \frac{3}{5} \right)^{2k} \left( \frac{2}{5} \right) = \left( \frac{2}{5} \right) \left( \frac{9/25}{1 - 9/25} \right) = \frac{9}{40}
$$

Putting this together, 
$$
P(X \text{ is even} | Y = 1) = \frac{9/40}{21/25} = \frac{15}{56}
$$
(e) (5 points) Find the marginal probability function of $X$.

Solution: $X = 1$ if the first ball is blue or the sequence is red/blue, so

$$P(X = 1) = \frac{2}{5} \cdot \left(\frac{3}{5}\right) \cdot \left(\frac{2}{5}\right) = \frac{16}{25}.$$ 

Otherwise for $x > 1$, $P(X = x) = P(X = x, Y = 1)$, so the distribution is

$$P(X = x) = \begin{cases} 
\frac{16}{25} & \text{if } x = 1 \\
\frac{3}{5}^x \left(\frac{2}{5}\right) & \text{if } x = 2, 3, 4, \ldots \\
0 & \text{otherwise}
\end{cases}$$

Note that this is almost the Geometric($2/5$) distribution, except that the geometric probabilities of 0 and 1 are combined here as $P(X = 1)$.

(f) (4 points) Find the marginal probability function of $Z$.

Solution: There are either $z - 1$ red balls and one blue ball or $z - 1$ blue balls and one red ball.

$$P(Z = z) = \left(\frac{3}{5}\right)^{z-1} \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^{z-1} \left(\frac{3}{5}\right), \ z = 2, 3, 4, \ldots$$