1. Bucket 1 contains one red ball, one green ball, and one blue ball. Bucket 2 contains two red balls and one blue ball. One of the two buckets is chosen at random, but Bucket 1 is two times as likely to be chosen as Bucket 2. One ball is selected from the chosen bucket at random (each ball in the chosen bucket is equally likely to be selected). A second ball is then chosen uniformly at random from the remaining balls in the chosen bucket.

Solution: It is probably helpful to draw a tree diagram to inform the calculations. Define the following events: $A = \{\text{Bucket 1 is chosen}\}$, $A^c = \{\text{Bucket 2 is chosen}\}$, $R_i = \{i\text{th ball is red}\}$, $B_i = \{i\text{th ball is blue}\}$, and $G_i = \{i\text{th ball is green}\}$. You should have determined that $\mathsf{P}(A) = 2/3$ and $\mathsf{P}(A^c) = 1/3$.

(a) (5 points) What is the probability that the first ball is red?

Solution: Apply the law of total probability, conditioning on the bucket.

$$\begin{aligned} \mathsf{P}(R_1) &= \mathsf{P}(A)\,\mathsf{P}(R_1 \mid A) + \mathsf{P}(A^c)\,\mathsf{P}(R_1 \mid A^c) \\ &= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \\ &= \frac{4}{9} \doteq 0.444. \end{aligned}$$

(b) (5 points) What is the probability that the second ball is green?

Solution: Law of total probability, condition on the bucket and the first ball, accounting for the fact that Bucket 2 has no green balls.

$$\begin{aligned} \mathsf{P}(G_2) &= \mathsf{P}(A)\,\mathsf{P}(G_1^c \mid A)\,\mathsf{P}(G_2 \mid A, G_1^c) + 0 \\ &= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{2}{9} \doteq 0.222. \end{aligned}$$

(c) (5 points) Given that the first ball is red, what is the probability that it was drawn from Bucket 1?

Solution: This is a Bayes' Theorem problem.

$$\begin{aligned} \mathsf{P}(A \mid R_1) &= \frac{\mathsf{P}(A)\,\mathsf{P}(R_1 \mid A)}{\mathsf{P}(R_1)} \\ &= \frac{(2/3)(1/3)}{4/9} \\ &= \frac{1}{2} = 0.5. \end{aligned}$$

(d) (5 points) Given that the first ball is red, what is the probability that the second ball is green?

Solution: This problem we can address using the definition of conditional probability and previous results.

$$\begin{aligned}
\mathsf{P}(G_2 \mid R_1) &= \frac{\mathsf{P}(R_1 \cap G_2)}{\mathsf{P}(R_1)} \\
&= \frac{\mathsf{P}(A \cap R_1 \cap G_2)}{\mathsf{P}(R_1)} \\
&= \frac{(2/3)(1/3)(1/2)}{4/9} \\
&= \frac{1}{4} = 0.25.
\end{aligned}$$

(e) (5 points) Are the events $A = \{\text{Bucket 1 is chosen}\}$ and $B = \{\text{first ball is blue}\}$ independent? Justify your response.

Solution: Events A and B are independent if and only if $\mathsf{P}(A \cap B) = \mathsf{P}(A)\,\mathsf{P}(B)$. Note that $\mathsf{P}(B \mid A) = \mathsf{P}(B \mid A^c) = 1/3$, and as these conditional probabilities are the equal to each other, the unconditional probability which is a weighted average of these must also be $1/3$ and the events must be independent. But here is the formal calculation.

$$\begin{aligned}
\mathsf{P}(A \cap B) &= \mathsf{P}(A)\,\mathsf{P}(B \mid A) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9} \\
\mathsf{P}(A) &= \frac{2}{3} \\
\mathsf{P}(B) &= \mathsf{P}(A)\,\mathsf{P}(B \mid A) + \mathsf{P}(A^c)\,\mathsf{P}(B \mid A^c) \\
&= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) \\
&= \frac{1}{3}
\end{aligned}$$

Thus, $\mathsf{P}(A \cap B) = \mathsf{P}(A)\,\mathsf{P}(B)$ and these events are independent.