

1. There are three fair 10-sided white dice with numbers  $0, 1, 2, \dots, 9$  and two fair regular 6-sided red dice with numbers  $1, 2, 3, 4, 5, 6$ . All five dice are rolled. The die rolls are mutually independent.

Solution: We can write the full sample space as  $S = S_1 \times S_2$  where  $S_1 = \{(w_1, w_2, w_3) : w_i = 0, 1, \dots, 9\}$  is the sample space for the white dice and  $S_2 = \{(r_1, r_2) : r_i = 1, 2, \dots, 6\}$  is the sample space for the red dice. Notice that  $|S_1| = 10^3 = 1000$ ,  $|S_2| = 6^2 = 36$ , and  $|S| = |S_1| \times |S_2| = 36,000$ . Some problems require consideration of the full sample space while others require only part.

- (a) (5 points) Find the probability that there are exactly two 9s.

Solution: As only the white dice may be 9s,  $P(\text{exactly two 9s}) = P(\text{exactly two white 9s})$ . There are  $\binom{3}{2} = 3$  ways to pick which two dice are 9s and nine choices for the remaining digit. Thus,

$$P(\text{exactly two white 9s}) = \frac{3 \times 9}{1000} = \frac{27}{1000} = 0.027.$$

- (b) (5 points) If each die roll is 4 or more, what is the probability that there are exactly two 9s.

Solution: Again, the results of the red dice are independent of the number of 9s, so we can consider outcomes in  $S_1$  alone. Using the definition of conditional probability, we derive the following:

$$\begin{aligned} P(\text{exactly two 9s} \mid \text{each die is 4 or more}) &= \frac{P(\text{exactly two 9s} \cap \text{each die is 4 or more})}{P(\text{each die is 4 or more})} \\ &= \frac{\binom{3}{2} 5 / 10^3}{6^3 / 10^3} \\ &= \frac{5}{72} \\ &\doteq 0.0694 \end{aligned}$$

- (c) (5 points) Find the probability that the maximum of all five dice is exactly 4.

Solution: Use the standard method of finding the probability that the maximum is less than or equal to some value. Notice that each white die has five ways to be 4 or lower while the red dice have only four ways each.

$$P(\max = 4) = P(\max \leq 4) - P(\max \leq 3) = \frac{5^3 \cdot 4^2 - 4^3 \cdot 3^2}{10^3 \cdot 6^2} = \frac{1424}{36,000} = \frac{89}{2250} \doteq 0.0396.$$

- (d) (5 points) Find the probability that the sum of the white dice is exactly 7.

Solution: This is a stars and bars problem with 7 \*s and two |s. All solutions are legal (do not put too many spots on any single die).

$$P(\text{sum of white dice is 7}) = \frac{\binom{9}{2}}{10^3} = \frac{36}{1000} = \frac{9}{250} = 0.036.$$

- (e) (5 points) Find the probability that the sum of all five dice is exactly 7.

Solution: The sum of the red dice is at least two, so we need to decide where the remaining five spots go. Again, all solutions are legal as putting them all on one of the red dice (plus the one already there) is just six. Hence, there are 5 \*s and 4 |s.

$$P(\text{sum of all dice is 7}) = \frac{\binom{9}{4}}{10^3 \cdot 6^2} = \frac{126}{36000} = \frac{7}{2000} = 0.0035.$$