Player A makes a series of independent bets, winning each with probability θ. When A wins a single bet, his fortune increases by c, when he loses, it decreases by c. His initial fortune is 3. He bets repeatedly until either he increases his fortune to 9 or he is ruined and his fortune is zero. If A reaches 9 before being ruined, we say A wins.

1. (5 points) If the size of each bet is $c = 1$ and if $\theta = 1/2$, find the probability that A wins.

   Solution:

$$\mathsf{P}\left(\text{A wins} \mid \theta = 1/2\right) = \frac{3}{9} = \frac{1}{3}$$

2. (5 points) If the size of each bet is $c = 1$ and if $\theta = 3/5$, find the probability that A wins.

   Solution:

$$\mathsf{P}\left(\text{A wins} \mid \theta = 3/5\right) = \frac{1 - \left(\frac{2/5}{3/5}\right)^3}{1 - \left(\frac{2/5}{3/5}\right)^9} \doteq 0.7225$$

3. (5 points) If the size of each bet is $c = 1$ and if $\theta = 2/3$, find the probability that A wins.

   Solution:

$$\mathsf{P}\left(\text{A wins} \mid \theta = 2/3\right) = \frac{1 - \left(\frac{1/3}{2/3}\right)^3}{1 - \left(\frac{1/3}{2/3}\right)^9} \doteq 0.8767$$

4. (5 points) Suppose that $\theta = (2 + X)/(4 + X)$ where $X \sim \text{Binomial}(2, 1/3)$ and the size of each bet is $c = 1$. Find the probability that A wins if θ is chosen at random. (Hint: *Use the Law of Total Probability.*)

   Solution:

$$\begin{aligned}
\mathsf{P}\left(\text{A wins}\right) &= \sum_{k=0}^{2} \mathsf{P}\left(X = k\right) \mathsf{P}\left(\text{A wins} \mid \theta = (2+k)/(4+k)\right) \\
&= \left(\frac{4}{9}\right) \mathsf{P}\left(\text{A wins} \mid \theta = 1/2\right) + \left(\frac{4}{9}\right) \mathsf{P}\left(\text{A wins} \mid \theta = 3/5\right) + \left(\frac{1}{9}\right) \mathsf{P}\left(\text{A wins} \mid \theta = 2/3\right) \\
&\doteq 0.5667
\end{aligned}$$

5. (5 points) If $\theta = 3/5$ and the bet size is $c = 3$, find the probability that A wins.

   Solution:

$$\mathsf{P}\left(\text{A wins} \mid \theta = 3/5, c = 3\right) = \frac{1 - \left(\frac{2/5}{3/5}\right)^1}{1 - \left(\frac{2/5}{3/5}\right)^3} \doteq 0.4737$$