1 Solutions for HW8

1:(a)Since X and Y are independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{24}xy^4e^{-(x+y)}$, $x > 0, y > 0$.
(b)$W > 0, 0 < Z < 1$
(c)$X = WZ$,$Y = W(1-Z)$,$|J| = W$,
$f_{W,Z}(w,z) = w\frac{1}{24}wzw^4(1-z)^4e^{-(wz+w(1-z))} = \frac{1}{24}w^6z(1-z)^4e^{-w}, w > 0, 0 < z < 1.$
(d)$f_W(w) = \int_0^1 f_{W,Z}(w,z)dz = \frac{1}{6!}w^6e^{-w}$, $w > 0$
$f_Z(z) = \int_0^\infty f_{W,Z}(w,z)dw = \frac{6!}{2!5!}z(1-z)^4$,$0 < z < 1$.
(e)Since $f_{W,Z}(w,z) = f_W(w)f_Z(z)$,W is independent of Z.

2:(a)Since X is independent of Y, $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$,$x \in R, y \in R$
(b)Let $X = R\cos\theta$,$y = R\sin\theta$, then $r > 0$,$\theta \in [0, 2\pi)$ and $|J| = r$,
so $f_{R,\theta}(r,\theta) = \frac{r}{2\pi}e^{-r^2/2}$,$r > 0$,
so $f_R(r) = \int_0^{2\pi} f_{R,\theta}(r,\theta)d\theta = re^{-r^2/2}$,$r > 0$.

3:(a) $Y|X \sim Bin(X, 1/3), X \sim Bin(5, 0.2)$, So

$$P(X = x, Y = y) = P(X = x)P(Y = y|X = x) = \binom{5}{x}0.2^x0.8^{4-x}\binom{x}{y}(1/3)^y(2/3)^{x-y}$$

$0 \le y \le x \le 5$

(b): the marginal of Y is
$\sum_y^5 P(X = x, Y = y)$, by calculation, $Y \sim Binomial(5, 1/15)$.

4:Note that for an exponentially distributed random variable with parameter λ, the c.d.f is $F_X(x) = 1 - e^{-\lambda x}$,
(a) $F_{x(1)}(x) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - e^{-\sum_{i=1}^n \lambda_i x}$,$x > 0$.
(b) $f_{x(1)}(x) = F'_{x(1)}(x) = (\sum_{i=1}^n \lambda_i)e^{\sum_{i=1}^n \lambda_i x}$, $x > 0$.
(c) $F_{x(n)}(x) = \prod_{i=1}^n P(X_i < x) = \prod_{i=1}^n (1 - e^{-\lambda_i x})$, $x > 0$.
(d) Since $\lambda_i = \lambda$, then $F_{x(n)}(x) = (1-e^{-\lambda x})^n$, $f_{x(n)}(x) = F'_{x(n)}(x) = n\lambda e^{-\lambda x}(1-e^{-\lambda x})^{n-1}$, $x > 0$.

5:(a) The joint distribution of X and Y is

	X=0	1	2	3
Y=0	$1/2^4$	$1/2^4$	0	0
1	$1/2^4$	$3/2^4$	$2/2^4$	0
2	0	$2/2^4$	$3/2^4$	$1/2^4$
3	0	0	$1/2^4$	$1/2^4$
$f_X(x)$	$2/2^4$	$6/2^4$	$6/2^4$	$2/2^4$

(b)The marginal density $P_{Y|X}(y|x) = \frac{P(X=x,Y=y)}{P(X=x)}$

So $P(Y=0|X=1) = 1/6$,$P(Y=1|X=1) = 1/2$,$P(Y=2|X=1) = 1/3$.

6:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}$$

Note: $x^2 - 2\rho xy + y^2 = (y-\rho x)^2 + (1-\rho^2)x^2$, so the marginal density of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(y-\rho x)^2+(1-\rho^2)x^2}{2(1-\rho^2)}} dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (\textbf{Note the integral part is the density of Norm}(\rho x, 1-\rho^2), So the integration is 1 \end{aligned}$$

$X \sim N(0,1)$, Similarly, we have $Y \sim N(0,1)$
"=>" We know X and Y are independent. So $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} * \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

So $\rho = 0$
"<=" , we have $\rho = 0$. So

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} = f_X(x)f_Y(y)$$

So X and Y are independent

7:(a)$F_{U_{(1)}}(u) = 1-(1-u)^n$,$u \in (0,1)$,
so $f_{U_{(1)}}(u) = F'_{U_{(1)}}(u) = n(1-u)^{(n-1)}$,$u \in (0,1)$.
(b)$F_{U_{(n)}}(u) = u^n$,$u \in (0,1)$,
so $f_{U_{(n)}}(u) = F'_{U_{(n)}}(u) = nu^{(n-1)}$,$u \in (0,1)$.
(c)$F_{U_{(1)},U_{(n)}}(x,y) = P(U_{(1)} \le x, U_{(n)} \le y) =$
when $x < 0$ or$y < 0$, 0
when $0 < x < 1$ and $y > 1$, $P(U_{(1)} < x) = 1-(1-x)^n$
when $0 < x < y < 1$, $P(U_{(n)} < y) - P(x < U_{(1)} < U_{(n)} < y) = y^n - (y-x)^n$
when $0 < y < 1$ and $x > y$, $P(U_{(n)} < y) = y^n$
when $x > 1$ and $y > 1$, 1.

(d) $f_{U_{(1)},U_{(n)}}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = n(n-1)(y-x)^{n-2}$,if $0 < x < y < 1$ and 0 otherwise.
8: (a)I_i is i.i.d Bernoulli distribution with $\theta = P(X_i \le x) = F(x)$. So W is Bin(n,F(x)).

(b)

$$P(W \geq k) = \sum_{i=k}^{n} P(W = i) = \sum_{i=k}^{n} (F(x))^i (1 - F(x))^{n-i}$$

(c)$W \geq k$ if and only if at least k X_is are less are equal x. It is equivalent to say $X_{(k)} \leq x$

$$G_k(x) = P(X_{(k)} \leq x) = P(W \geq k) = \sum_{i=k}^{n} (F(x))^i (1 - F(x))^{n-i}$$

(d)

$$\begin{aligned} g_k(x) &= \frac{dG_k(x)}{dx} \\ &= \sum_{i=k}^{n} [\binom{n}{i} i f(x)(F(x))^{i-1}(1 - F(x))^{n-i} - \binom{n}{i}(n-i) f(x)(F(x))^i (1 - F(x))^{n-i-1}] \end{aligned}$$

(e) So $a_i = i\binom{n}{i}, b_i = (n-i)\binom{n}{i}$. So $b_n = 0$

$$a_{j+1} = (j+1)\frac{n!}{(j+1)!(n-j-1)!} = \frac{n!}{j!(n-j-1)!}$$

$$b_j = (n-j)\frac{n!}{j!(n-j)!} = \frac{n!}{j!(n-j-1)!}$$

So $a_{j+1} = b_j$.

$$g_k(x) = \frac{n!}{(n-k)!(k-1)!} f(x)(F(x))^{k-1}(1 - F(x))^{n-k}$$

(f) If $X_i \sim Unifrom[0, 1]$, then $f(x) = 1, F(x) = x, x \in (0, 1)$. So

$$g_k(x) = \frac{n!}{(n-k)!(k-1)!}(x)^{k-1}(1 - x)^{n-k}, x \in (0, 1)$$

For Beta(k,n-k+1), the density is

$$f(x) = \frac{x^{k-1}(1-x)^{n-k}}{B(k, n-k+1)}$$

$B(k, n-k+1) = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n+1)} = \frac{(k-1)!(n-k)!}{n!}$. So $g_k(x) = f(x)$. $X_{(k)} \sim Beta(k, n-k+1)$

9:(a)$F(x) = \int_{-\infty}^{x} f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$, $x \in R$.

(b) $F^{-1}(y) = tan(\pi y - \frac{\pi}{2})$,$0 \leq y \leq 1$

(c) $X = F^{-1}(U) = tan(\pi y - \frac{\pi}{2})$, where $U \sim Uniform(0, 1)$.

Since F is the c.d.f of Cauchy distribution and U is uniform between 0 and 1, X has the cauchy distribution.