1 Solutions for HW6

1: (a) $P(U \leq 1) = 1$
(b)P(U=1/2)=0
(c)$P(U^2 < 1/4) = P(-1/\sqrt{4} < U < 1/\sqrt{4}) = \int_0^{1/2} 1dx = 1/2$
(d)$P(|2U-1| > 1/2) = P(2U-1 > 1/2 or 2U-1 < -1/2) = P(2U-1 > 1/2) + P(2U-1 < -1/2) = 1 - \int_{1/4}^{3/4} 1dx = 1/2$
(e)$P(|U - 1/2| > 0.3) = P(U > 0.7 or U < 0.3) = P(0 \leq U < 0.2) + P(U > 0.8) = 0.2 + \int_{0.8}^{1} 1dx = 0.2 + 0.2 = 0.4$

2: For Exp(λ), $P(X \geq x) = e^{-\lambda x}$ for $x > 0$
(a) $P(W \geq 2) = e^{-16}$
(b) $P(W \leq 10) = 1 - e^{-8*10} = 1 - e^{-80}$
(c) $P(1 < W < 5) = P(W > 1) - P(W > 5) = e^{-8} - e^{-40}$
(d) $P(W^2 < 4) = P(-2 < W < 2) = P(0 \leq W < 2) = 1 - P(W \geq 2) = 1 - e^{-16}$
(e) $P(W > a) = e^{-8a}$

3: For the density $f(x)$, $\int_{-\infty}^{\infty} f(x)dx = 1$
(a)$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 c_1 x^3 dx = c_1/4 = 1$. So $c_1 = 4$.

$$P(X_1 < 1/2) = \int_0^{1/2} 4x^3 dx = (1/2)^4$$

(b)$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 c_2 x(2-x)dx = c_2 4/3 = 1$, So $c_2 = 3/4$

$$P(X_2 > 1/2) = 1 - \int_0^{1/2} \frac{3}{4} x(2-x)dx = 27/32 = .084375$$

(c)$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} c_3 \frac{1}{1+x^2} dx = c_3 arctan(x)|_{-\infty}^{\infty} = c_3 \pi = 1$, so $c_3 = 1/\pi$

$$P(X_3 < 1/2) = \int_{\infty}^{1/2} c_3 \frac{1}{(1+x^2)\pi} = \frac{arctan(x)|_{-\infty}^{1/2}}{\pi} = \frac{arctan(1/2) + \pi/2}{\pi}$$

(d)$\int_0^4 c_4(x^3 - 3x^2 + 2x + 1)dx = 20c_4 = 1$, so $c_4 = 1/20$

$$P(2 < X_4 < 3) = \int_2^3 (x^3 - 3x^2 + 2x + 1)/20 dx = 13/80 = .1625$$

4: Take $x = rcos\theta, y = rsin\theta$, then the Jaccobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} cos\theta & sin\theta \\ -rsin\theta & rcos\theta \end{vmatrix} = r$$

$$\begin{aligned} C^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}dxdy \\ &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2/2}rdrd\theta \\ &= \int_{0}^{2\pi} 1d\theta \\ &= 2\pi \end{aligned}$$

So $C = \sqrt{2\pi}$

5:(a) $\int_0^2 c(2x^2 - x^3)dx = 4c/3 = 1$, so c=3/4

(b)

$$F(x) = \begin{cases} 0, x < 0 \\ \int_0^x \frac{3}{4}(2t^2 - t^3)dt = \frac{1}{2}x^3 - \frac{3}{16}x^4, 0 \le x < 2 \\ 1, x \ge 2 \end{cases}$$

(c) $P(0.5 < X < 1.5) = F(1.5) - F(0.5) = 11/16 = 0.6875$

6: (a) $P(Z > 1) = P(Z < -1) = \Phi(-1) = 0.1587$

(b)$P(Z > -1.49) = 1 - \Phi(-1.49) = 1 - 0.0681 = 0.9319$

(c)$P(-1.96 < Z < 1.96) = \Phi(1.96) - \Phi(-1.96) = 1 - 2\Phi(-1.96) = 1 - 2 * 0.025 = 0.95$

(d)$P(Z > z) = P(Z < -z) = \Phi(-z) = 0$, So $z = 0$

(e)$P(|Z| > 2.33) = 2P(Z < -2.33) = 2 * \Phi(-2.33) = 2 * 0.0099 = 0.0198$

(f)$P(|Z| < z) = P(-z < Z < z) = \Phi(z) - \Phi(-z) = 1 - 2\Phi(-z) = 0.9$, so $\Phi(-z) = 0.05, -z = -1.645, z = 1.645$

7:Let $Z = (Y - 200)/25, Z \sim N(0, 1)$

(a) $P(Y > 205) = P((Y - 200)/25 > (205 - 200)/25) = P(Z > 0.2) = 0.4207$ (b)$P(Y > 192) = P(Z > (192 - 200)/25) = P(Z > -0.32) = 1 - \Phi(-0.32) = 0.6255$

(c)$P(180 < Y < 220) = P(-0.8 < Z < 0.8) = \Phi(0.8) - \Phi(-0.8) = 0.5762$

(d)$P(Z > (y - 200)/25) = 0.2$, (y-200)/25=0.845, y=221

(e)$P(|Y - 200| > 40) = P(|Z| > 1.6) = 2 * \Phi(-1.6) = 0.1096$

(f)$P(|Y - 200| < y) = P(|Z| < y/25) = 0.9$, we have y/25=1.645, so y=41

8: $F(x) = \int_{\infty}^{x} \frac{1}{\pi(1+t^2)}dt = \frac{arctan(x)+\pi/2}{\pi}$

$P(X > 1/\sqrt(3)) = 1 - F(1/\sqrt(3)) = 1 - (1/6 + 1/2) = 1/3$