Chapter 4 Summary

Sampling Distributions

- If $X_1, \ldots, X_n$ is an i.i.d. sample from some distribution and $Y = h(X_1, \ldots, X_n)$ is a random variable determined from the sample by some function $h$, then the distribution of $Y$ is called a sampling distribution.

Convergence of Random Variables

- Three notions of convergence of random variables:
  - $X_n \xrightarrow{a.s.} Y$ or $X_n$ converges to $Y$ almost surely or with probability one as $n \to \infty$ if
    $$
P \left( \lim_{n \to \infty} X_n = Y \right) = 1.
    $$
  - $X_n \xrightarrow{p} Y$ or $X_n$ converges to $Y$ in probability as $n \to \infty$ if
    $$
    \lim_{n \to \infty} P \left( |X_n - Y| \geq \epsilon \right) = 0
    $$
    for any $\epsilon > 0$.
  - $X_n \xrightarrow{D} Y$ or $X_n$ converges to $Y$ in distribution as $n \to \infty$ if
    $$
    \lim_{n \to \infty} F_{X_n}(x) = F_Y(x)
    $$
    for all $x$ such that $P(Y = x) = 0$ where $F_Z$ is the cumulative distribution function of $Z$.

- These notions of convergence are not identical.
  - $X_n \xrightarrow{a.s.} Y \Rightarrow X_n \xrightarrow{p} Y$ as $n \to \infty$ but the converse is not true in general.
  - $X_n \xrightarrow{p} Y \Rightarrow X_n \xrightarrow{D} Y$ as $n \to \infty$ but the converse is not true in general.

- If $m_Z(s)$ is the moment generating function of random variable $Z$ then
  $$
  \lim_{n \to \infty} m_{X_n}(s) = m_Y(s) \Rightarrow X_n \xrightarrow{D} Y
  $$
  provided the moment generating functions exist.

Laws of Large Numbers

- If $X_1, \ldots, X_n$ is an i.i.d. sample from a distribution with finite mean $\mu = \mathbb{E}(X_1)$, then $M_n = (X_1 + \cdots + X_n)/n$ is the sample mean and $\mathbb{E}(M_n) = \mu$.

- The weak law of large numbers says that if we also have $\text{Var}(X_1) = \sigma^2$, then $M_n \xrightarrow{p} \mu$.

- The strong law of large numbers says that $M_n \xrightarrow{a.s.} \mu$.

Central Limit Theorem

- If $X_1, \ldots, X_n$ is an i.i.d. sample, $\mathbb{E}(X_i) = \mu, \text{Var}(X_i) = \sigma^2 < \infty, S_n = \sum_{i=1}^{n} X_i$, and $M_n = S_n/n$.
- $\mathbb{E}(S_n) = n\mu, \text{Var}(S_n) = n\sigma^2$.
- $\mathbb{E}(M_n) = \mu, \text{Var}(M_n) = \sigma^2/n$.
- The standardized random variable (mean 0, variance 1) is
  $$
  Z_n = \frac{S_n - n\mu}{\sigma \sqrt{n}} = \frac{M_n - \mu}{\sigma \sqrt{n}}.
  $$
- If $Z \sim N(0,1)$, then $Z_n \xrightarrow{D} Z$ as $n \to \infty$.
- For sufficiently large $n$,
  $$
  \mathbb{P}(S_n \leq c) = \mathbb{P} \left( Z_n \leq \frac{c - n\mu}{\sigma \sqrt{n}} \right) \approx \Phi \left( \frac{c - n\mu}{\sigma \sqrt{n}} \right) .
  $$
- For sufficiently large $n$,
  $$
  \mathbb{P}(M_n \leq c) = \mathbb{P} \left( Z_n \leq \frac{c - \mu}{\sigma \sqrt{n}} \right) \approx \Phi \left( \frac{c - \mu}{\sigma \sqrt{n}} \right) .
  $$

Distributions Based on the Normal

- Sums of independent normal random variables are normal. If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$ and $Y = \sum_{i=1}^{n} X_i$,
  $$
  Y \sim N \left( \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2 \right).
  $$
- $\chi^2$ distributions: If $Z_1, \ldots, Z_n \sim$ i.i.d. $N(0,1)$ and $X = Z_1^2 + \cdots + Z_n^2$, then we say $X \sim \chi^2(n)$.
  - The $\chi^2(n)$ distribution and the Gamma($n/2, 1/2$) distributions are identical.
  - $f(x) = (1/2)^{n/2} \pi^{n/2} 2^{1-x} e^{-x/2}$ for $x > 0$
  - $\mathbb{E}(X) = n, \text{Var}(X) = 2n$.
- $t$ distributions: If $Z \sim N(0,1), Y \sim \chi^2(n)$, and $Z$ and $Z$ are independent, then $X = Z/\sqrt{Y/n} \sim t(n)$.
  - The $t(1)$ distribution is known also as the Cauchy distribution.
  - $f(x) = \Gamma(1/2) / \Gamma(1/2)^2 \sqrt{n} \left( 1 + (\frac{x^2}{n}) \right)^{-1/2}$ for $x \in \mathbb{R}$
- $\mathbb{E}(X) = 0$ for $n > 1, \text{Var}(X) = n/(n-2)$ for $n > 2$.
- $F$ distributions: If $W \sim \chi^2(m), Y \sim \chi^2(n)$, and $W$ and $Y$ are independent, then $X = \frac{W/m}{Y/n} \sim F(m, n)$.
  - $f(x) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2) \Gamma(n/2)} \left( \frac{m}{n} \right)^{m+1} x^{m} \left( 1 + \frac{mx}{n} \right)^{-(m+n)/2}$ for $x > 0$.
  - $\mathbb{E}(X) = n/(n-2)$ for $n > 2$
  - $\text{Var}(X) = 2n^2(m+n-2)/(m(n-2)^2(n-4))$ for $n > 4$.  
