Statistics/Mathematics 309 — Inclusion-Exclusion Proof
Fall 2009

Here is a clean version of a proof of the Inclusion-Exclusion theorem.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We begin by showing that $A \cup B = A \cup (A^c \cap B)$. Notice that this simply says that $A \cup B$ is the set of everything in A and everything in B that is not also in A. Beginning with the right-hand side,

$$\begin{aligned} A \cup (A^c \cap B) &= (A \cup A^c) \cap (A \cup B) \quad \text{by set algebra} \\ &= S \cap (A \cup B) \quad \text{since } A \text{ and } A^c \text{ form a partition of } S \\ &= A \cup B \quad \text{since } (A \cup B) \subset S \end{aligned}$$

It follows then that

$$P(A \cup B) = P(A) + P(A^c \cap B)$$

by an axiom of probability as A and $(A^c \cap B)$ are disjoint.

So, we need to show that $P(A^c \cap B) = P(B) - P(A \cap B)$ to finish. We begin by showing that $B = (A \cap B) \cup (A^c \cap B)$. Again, this should be clear since B can be partitioned into its intersections with A and A^c, but we can again show this formally.

$$\begin{aligned} (A \cap B) \cup (A^c \cap B) &= (A \cup A^c) \cap B \quad \text{by set algebra} \\ &= S \cap B \quad \text{since } A \cup A^c = S \\ &= B \quad \text{since } B \subset S \end{aligned}$$

Since $(A \cap B)$ and $(A^c \cap B)$ are disjoint, the axioms of probability allow us to conclude that

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

which we can rearrange algebraically as

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

and the proof is complete.