Here is a clean version of a proof of the Inclusion-Exclusion theorem.

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

We begin by showing that \( A \cup B = A \cup (A^c \cap B) \). Notice that this simply says that \( A \cup B \) is the set of everything in \( A \) and everything in \( B \) that is not also in \( A \). Beginning with the right-hand side,

\[
A \cup (A^c \cap B) = (A \cup A^c) \cap (A \cup B) \quad \text{by set algebra}
\]

\[
= S \cap (A \cup B) \quad \text{since } A \text{ and } A^c \text{ form a partition of } S
\]

\[
= A \cup B \quad \text{since } (A \cup B) \subset S
\]

It follows then that

\[ P(A \cup B) = P(A) + P(A^c \cap B) \]

by an axiom of probability as \( A \) and \((A^c \cap B)\) are disjoint.

So, we need to show that \( P(A^c \cap B) = P(B) - P(A \cap B) \) to finish. We begin by showing that

\[ B = (A \cap B) \cup (A^c \cap B) \]

Again, this should be clear since \( B \) can be partitioned into its intersections with \( A \) and \( A^c \), but we can again show this formally.

\[
(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B \quad \text{by set algebra}
\]

\[
= S \cap B \quad \text{since } A \cup A^c = S
\]

\[
= B \quad \text{since } B \subset S
\]

Since \((A \cap B)\) and \((A^c \cap B)\) are disjoint, the axioms of probability allow us to conclude that

\[ P(B) = P(A \cap B) + P(A^c \cap B) \]

which we can rearrange algebraically as

\[ P(A^c \cap B) = P(B) - P(A \cap B) \]

and the proof is complete.