Assignment #8 — Due Wednesday, October 28, 2009, by 5:00 P.M.

Turn in homework in lecture, discussion, or your TA's mailbox (just inside the main entrance of MSC, 1300 University Avenue) Please circle the discussion section you expect to attend to pick up this assignment.

311: Monday 1:20–2:10 **312:** Monday 12:05–12:55 **313:** Tuesday 8:25–9:15

1. Suppose that $X \sim \text{Gamma}(2, 1)$ and $Y \sim \text{Gamma}(5, 1)$ and that X and Y are independent.
   (a) Find the joint density $f_{X,Y}$. Draw a sketch of the portion of $\mathbb{R}^2$ where this joint density is positive.
   (b) If $W = X + Y$ and $Z = X/(X + Y)$, draw a sketch of the portion of $\mathbb{R}^2$ where the joint density $f_{W,Z}$ is positive.
   (c) Find the joint density $f_{W,Z}$.
   (d) Find the marginal densities of W and of Z. If these are examples of named densities, identify them and the associated parameter values.
   (e) Are W and Z independent? Explain.
2. Suppose that X and Y are independent $\text{N}(0, 1)$ random variables. Find the density of $R = \sqrt{X^2 + Y^2}$ which is the distance of the pair (X, Y) from the origin. *(Hint: use polar coordinates for the change of variable and introduce a second random variable.)*
3. Suppose that $X \sim \text{Binomial}(5, 0.2)$ and Y is the number of 1's or 2's in X rolls of a fair six-sided die. (If $X = 0$, then $Y = 0$.)
   (a) Find the joint probability function for X and Y.
   (b) Find the marginal distribution of Y.
4. Assume that $X_i \sim \text{Exponential}(\lambda_i)$ for $i = 1, \ldots, n$ and that these random variables are mutually independent.
   (a) Find the cdf of $X_{(1)} = \min_i X_i$.
   (b) Find the density of $X_{(1)}$ by differentiating the cdf.
   (c) Find the cdf of $X_{(n)} = \max_i X_i$.
   (d) Find the density of $X_{(n)}$ by differentiating the cdf.
5. A fair coin is tossed four times in succession. X is the number of heads in the first three coin tosses. Y is the number of heads in the last three coin tosses.
   (a) Find the joint probability function for X and Y.
   (b) Find the conditional probability function $p_{Y\,|\,X}(y\,|\,x)$ for $x = 1$.
6. Exercise 2.8.17. Let X and Y have the bivariate normal distribution as described in Example 2.7.9 on page 85, except let $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Prove that X and Y are independent if and only if $\rho = 0$. (In other words, show that if $\rho = 0$, then $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \phi(x)\phi(y)$ and also show that if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, then it must be the case that $\rho = 0$. For the second part, you need to find the marginal densities of X and Y.)

7. Suppose that $U_1, \ldots, U_n$ are an iid (independent and identically distributed) sample from the Uniform$(0,1)$ distribution. Let $\min_i(U_1, \ldots, U_n) = U_{(1)} < U_{(2)} < \cdots < U_{(n)} = \max_i(U_1, \ldots, U_n)$ be the *order statistics* of the sample.

   (a) Find the density of $U_{(1)}$ using the cdf method. Which named distribution is this?

   (b) Find the density of $U_{(n)}$ using the cdf method. Which named distribution is this?

   (c) Find the joint cdf of $U_{(1)}$ and $U_{(n)}$. Specifically, for all $(x, y) \in \mathbb{R}^2$, find a formula for $F(x, y) = P\left(U_{(1)} < x \cap U_{(n)} < y\right)$. *(Hint: there are special cases when the point (x, y) is not in the unit square, and different cases when $x < y$ and $x > y$ within the unit square.)*

   (d) It turns out that the joint density can be found from the joint cdf by differentiating for each variable. Namely,
   $$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \, \partial y} \; .$$
   This is the natural extension of the relationship between the cdf and densities for single random variables. Use this fact to find the joint density of $U_{(1)}$ and $U_{(n)}$.

8. This exercise will step you through a derivation of the density of the *order statistic* $X_{(k)}$, the kth smallest value of an i.i.d. sample $X_1, \ldots, X_n$ where X_i is an absolutely continuous random variable with cdf F and density $f = F'$. The method will be to find the cdf of $X_{(k)}$ and take its derivative.

   (a) For a fixed value x such that $0 < F(x) < 1$, let I_i be an indicator random variable of the event $\{X_i \le x\}$. What is the (named) distribution of $W = I_1 + \cdots + I_n$?

   (b) Write an expression that is a sum of probabilities for the event $\{W \ge k\}$.

   (c) Explain in words why the events $\{W \ge k\}$ and $\{X_{(k)} \le x\}$ are identical. Thus, if $G_k(x) = P\left(X_{(k)} \le x\right)$ is the cdf of $X_{(k)}$, then $G_k(x)$ is equal to the expression you found in (b).

   (d) Let g_k be the density of $X_{(k)}$ so that $g_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}(G_k(x))$. Recalling that the derivative of a sum is the sum of the derivatives, express $g_k(x)$ as a sum. This sum should be of the form
   $$\sum_{i=k}^{n} \Big(a_i f(x)(F(x))^{i-1}(1 - F(x))^{n-i} - b_i f(x)(F(x))^{i}(1 - F(x))^{n-i-1} \Big)$$
   for some constants (expressions that do not depend on x) $\{a_i\}$ and $\{b_i\}$.

   (e) This sum can be broken into a sum of positive terms (with a_i) and a sum of negative terms (with b_i). Notice that the first sum takes the form
   $$a_k f(x)(F(x))^{k-1}(1 - F(x))^{n-k} + a_{k+1} f(x)(F(x))^{k}(1 - F(x))^{n-k-1} + \cdots + a_n f(x)(F(x))^{n-1}(1 - F(x))^{0}$$
   and that the second sum takes the form
   $$-b_k f(x)(F(x))^{k}(1 - F(x))^{n-k-1} - \cdots - b_{n-1} f(x)(F(x))^{n-1}(1 - F(x))^{0} - b_n f(x)(F(x))^{n}(1 - F(x))^{-1}$$
   so that the sums can be recombined by matching terms with common exponents and written as
   $$a_k f(x)(F(x))^{k-1}(1 - F(x))^{n-k} + \left(\sum_{j=k}^{n-1} (a_{j+1} - b_j) f(x)(F(x))^{j}(1 - F(x))^{n-j-1} \right) - b_n f(x)(F(x))^{n}(1 - F(x))^{-1} \; .$$
   Show that $a_{j+1} = b_j$ and that $b_n = 0$ so that the density of the order statistic $X_{(k)}$ is
   $$g_k(x) = \frac{n!}{(n-k)!(k-1)!} f(x)(F(x))^{k-1}(1 - F(x))^{n-k} \; .$$

9. Let X have the Cauchy distribution with density $1/(\pi(1+x^2))$ for $-\infty < x < \infty$.

   (a) Find the cdf of X.

   (b) Find the quantile function (inverse cdf) of X.

   (c) Find a function $X = g(U)$ where $U \sim \text{Uniform}(0, 1)$ so that X has the Cauchy distribution.

Work to do, but not turn in.

- Read Chapter 2, remainder.