

Assignment #4 — Due Wednesday, September 30, 2009, by 5:00 P.M.

Turn in homework in lecture, discussion, or your TA's mailbox (just inside the main entrance of MSC, 1300 University Avenue) Please circle the discussion section you expect to attend to pick up this assignment.

311: Monday 1:20–2:10**312:** Monday 12:05–12:55**313:** Tuesday 8:25–9:15

1. A 6-sided die is specially constructed so that the probability of rolling a 1 or 6 is each $1/4$ and the probability of a 2, 3, 4, or 5 is each $1/8$. Define $A = \{2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$. Find an event C such that $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but events A , B , and C are **not mutually independent**.
 2. An urn contains two black balls and three white balls. One ball is drawn uniformly at random. The ball is replaced in the urn along with another ball of the same color. A second ball is then drawn from the urn. Given that the second ball drawn is white, what is the probability that the first ball drawn was also white?
 3. An urn contains 100 numbered balls. Nine balls each are numbered 1, 2, and 3, and the remainder are numbered either 4 or 5. One ball is drawn uniformly at random.
Define these events: $A = \{1, 2, 4\}$, $B = \{1, 3, 4\}$, and $C = \{2, 3, 4\}$.
 - (a) If events A , B , and C are pairwise independent, what numbers are on the remaining 73 balls?
 - (b) Are events A , B , and C mutually independent?
 4. Let $X \sim \text{Geometric}(\theta)$. Find $P(2 \leq X \leq 12)$.
 5. Let $X \sim \text{Binomial}(8, \theta)$. For what value of θ is $P(X = 6)$ maximized?
 6. Let $X \sim \text{Poisson}(\lambda)$. For what value of λ is $P(X = 6)$ maximized?
 7. An urn contains 2 red balls and 8 blue balls. When balls are drawn at random, each ball is equally likely to be selected. Balls may or may not be replaced after each draw as specified in each specific problem. For each part, identify the named distribution of the random variable and any associated parameters (if applicable) in addition to the computing the probability.
 - (a) Six balls are drawn at random, with replacement. Find the probability that exactly two of the balls drawn are red.
 - (b) Six balls are drawn at random, without replacement. Find the probability that exactly two of the balls drawn are red.
 - (c) Balls are drawn repeatedly with replacement. Find the probability that exactly four blue balls appear before the second red ball.
 - (d) Balls are drawn repeatedly with replacement. Find the probability that exactly four red balls appear before the second blue ball.
 - (e) Balls are drawn repeatedly with replacement. Find the probability that two or more blue balls appear before the first red ball.
 - (f) Balls are drawn repeatedly without replacement. Find the probability that two or more blue balls appear before the first red ball.
 8. Events A and B have indicator random variables I_A and I_B respectively. Define random variable $X = I_A \times I_B$. Is X an indicator random variable? If so, then of what event?
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9. Suppose that X is a random variable and we define $Y = \log(X - 1)$. Under what conditions is Y a random variable and when is it not?
10. A fair 4-sided die with sides 0, 0, 2, 3 is rolled three times. Define X_i to be the value of the i th roll for $i = 1, 2, 3$.
- Find $P(X_1 = x)$ for all real x .
 - Let $Y = X_1 + X_2 + X_3$. Find $P(Y = y)$ for all real y .
 - Define $Z = X_1 \times X_2 \times X_3$. Find $P(Z = z)$ for all real z .
11. Suppose that $X \sim \text{Binomial}(m, \theta)$ and that $Y \sim \text{Binomial}(n, \theta)$ and that X and Y are independent (meaning that $P(X = x \cap Y = y) = P(X = x)P(Y = y)$ for all x, y). Let $Z = X + Y$. Verify that $Z \sim \text{Binomial}(m + n, \theta)$ by simplifying the expression

$$P(Z = z) = \sum_{k=0}^z P(X = k \cap Y = z - k)$$

algebraically.

You may use this fact:

$$\sum_k \binom{m}{k} \binom{n}{z-k} = \binom{m+n}{z}$$

where the sum ranges over k for which the binomial coefficients are defined. This fact can be proven by a combinatorial bijection: the right hand side counts the number of ways to choose z objects from $m + n$ objects. The left hand side does the same, but partitions the count by the number of objects that come from each of two types, and sums over the partition.

12. A fair coin is tossed ten times. Let X be the number of heads and Y be the number of tails. What distributions do X and Y have? Let $Z = X + Y$. Is $Z \sim \text{Binomial}(20, 0.5)$? Explain why or why not. How does your answer relate to the previous problem?

Work to do, but not turn in.

- Read Chapter 2, remainder.