Correction for Change of Variable Problem

I made an error in class in finding the distribution of Y when $X \sim \mathrm{N}(0,1)$ and $Y = X^2$. Here are two correct solutions.

Using the CDF. The idea here is to find the cdf of Y and take its derivative to find the density. First note that $F_Y(y) = 0$ if $y < 0$, so assume that $y \geq 0$ below.

$$\begin{aligned}
F_Y(y) &= \mathsf{P}\left(Y \leq y\right) \\
&= \mathsf{P}\left(X^2 \leq y\right) \\
&= \mathsf{P}\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) \qquad \text{(I erred with } \mathsf{P}\left(X \leq \sqrt{y}\right) \text{ in class.)} \\
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})
\end{aligned}$$

Notice that since we are about to take the derivative, there is no point in evaluating the integral (which we cannot do here anyway). Recall the calculus chain rule $\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(g(x))g'(x)$ and also that $\phi(z) = \phi(-z) = \mathrm{e}^{-z^2/2}/\sqrt{2\pi}$.

$$\begin{aligned}
f_Y(y) &= \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} \\
&= \frac{\mathrm{d}(\Phi(\sqrt{y}) - \Phi(-\sqrt{y}))}{\mathrm{d}y} \\
&= \phi(\sqrt{y})\frac{1}{2\sqrt{y}} - \phi(-\sqrt{y})\frac{-1}{2\sqrt{y}} \\
&= \frac{\phi(\sqrt{y})}{\sqrt{y}} \\
&= \frac{\mathrm{e}^{-y/2}}{\sqrt{2\pi y}}, \quad y > 0
\end{aligned}$$

On Wednesday, I was off by a factor of (1/2). It may not be immediately obvious, but this is one of our friendly named distributions. Recalling that $\Gamma(1/2) = \sqrt{\pi}$, we could rewrite the density as

$$f_Y(y) = \frac{(1/2)^{(1/2)}}{\Gamma(1/2)} y^{(1/2)-1}\mathrm{e}^{-(1/2)y}, \quad y > 0$$

and see that $Y \sim \mathrm{Gamma}(1/2, 1/2)$ which we will learn in a later chapter is also known as the $\chi^2(1)$ distribution.

Using the formula. Even though $h(x) = x^2$ is not one-to-one over the whole real line, we can break it into one-to-one intervals where $x \in (-\infty, 0)$ or $x \in [0, \infty)$. For the first interval, $x = h^{-1}(y) = -\sqrt{y}$ and in the second interval $x = h^{-1}(y) = \sqrt{y}$. The density at y is the sum of the contributions to the density from all points x such that $h(x) = y$. In this case, each y (except $y = 0$) receives a contribution from two x. So, we can find the density of Y by direct use of the formula from class. Notice that $h'(x) = 2x$ and we replace x by the two possible inverses in turn.

$$\begin{aligned}
f_Y(y) &= \frac{1}{\left|2(-\sqrt{y})\right|}\phi(-\sqrt{y}) + \frac{1}{\left|2(\sqrt{y})\right|}\phi(\sqrt{y}) \\
&= \frac{\phi(\sqrt{y})}{\sqrt{y}} \\
&= \frac{\mathrm{e}^{-y/2}}{\sqrt{2\pi y}}, \quad y > 0
\end{aligned}$$

This agrees with the previous derivation.