# Solutions for Homework 14

1: (a) 
\[ P(\tau_a < \tau_b) = P(\tau_0 < \tau_{a+b}) = 1 - \frac{a}{a+b} = \frac{b}{a+b} \]

(b) 
\[ \lim_{b \to \infty} P(\tau_a < \tau_b) = P(\tau_a < \infty) = P(\text{ever hit -a}) = \lim_{b \to \infty} \frac{b}{a+b} = 1 \]

(c) 
\[ P(\tau_a > \tau_b) = 1 - P(\tau_a < \tau_b) = \frac{a}{a+b} \]
\[ \lim_{-a \to -\infty} P(\tau_a > \tau_b) = P(\tau_b < \infty) = P(\text{ever hit b}) = \lim_{a \to \infty} \frac{a}{a+b} \]

(d) From (b) and (c), for any negative integer -a, P(\text{ever hit -a})=1; for any positive integer b P(\text{ever hit b})=1. So P(A=Z)=1

2: (a) 
\[ q/p = \frac{(2/3)}{(1/3)} = 2 \]
\[ P(\tau_a > \tau_b) = P(\tau_0 > \tau_{a+b}) = \frac{1 - (q/p)^a}{1 - (q/p)^{a+b}} = \frac{1 - 2^a}{1 - 2^{a+b}} \]

(b) 
\[ \lim_{-a \to -\infty} P(\tau_a > \tau_b) = P(\tau_b < \infty) = P(\text{ever hit b}) = \lim_{a \to \infty} \frac{1 - 2^a}{1 - 2^{a+b}} = \frac{1}{2^b} \]

(c) 
\[ P(M \leq m) = P(X_n \text{ never hits } m+1) = 1 - P(\text{ever hit } m+1) = 1 - \frac{1}{2^{m+1}} \]

(d) 
\[ P(M = m) = P(M \leq m) - P(M \leq m - 1) = 1 - \frac{1}{2^{m+1}} - \left(1 - \frac{1}{2^m}\right) = \frac{1}{2^{m+1}} \]

(e) 
\[ E(M) = \sum_{m=0}^{\infty} P(X > m) = \sum_{m=0}^{\infty} \frac{1}{2^k+1} = \frac{1/2}{1 - 1/2} = 1 \]

3: (a) Neglect, a 16*16 matrix
(b) \[
P(X_2 = k | X_0 = 2) = \begin{cases} 
\frac{1}{9} + \frac{1}{9} + \frac{1}{12} = \frac{11}{36} & k = 2 \\
(1/3) \ast (1/4) = 1/12 & k = 4 \\
1/12 & k = 5 \\
1/9 & k = 7 \\
1/9 & k = 10 \\
1/12 & k = 13 \\
2/9 & k = 15 
\end{cases}
\]

(c) The periodicity is 2 for every state.

(d) It is irreducible. Every state can go to any other states.

(e) Guess the probability is \((2, 3, 3, 2, 3, 4, 4, 3, 4, 3, 4, 3, 4, 3, 2)/48\)

It is easy to verify that it is the stationary distribution

4 it is easy to check \((1/8, 3/8, 2/8, 2/8)\) is the stationary distribution.

(a) from the equations
\[
\begin{align*}
f_{00} &= 1 + f_{10} \\
f_{10} &= 1 + \frac{1}{3}(f_{20} + f_{30}) \\
f_{20} &= 1 + \frac{1}{2}(f_{10} + f_{30}) \\
f_{30} &= 1 + \frac{1}{2}(f_{10} + f_{20})
\end{align*}
\]

Solve these equations, we have \(f_{00} = 8\)

(b) for \(f_{11}\), we have the following system of equations
\[
\begin{align*}
f_{11} &= \frac{1}{3}(1 + f_{01}) + \frac{1}{3}(1 + f_{21}) + \frac{1}{3}(1 + f_{31}) \\
f_{01} &= 1 \\
f_{21} &= \frac{1}{2} + \frac{1}{2}(1 + f_{31}) \\
f_{31} &= \frac{1}{2} + \frac{1}{2}(1 + f_{21})
\end{align*}
\]

Solve the equations, we have \(f_{11} = \frac{8}{7}\)

for \(f_{22}\), the system of equations are
\[
\begin{align*}
f_{22} &= \frac{1}{2}(1 + f_{12}) + \frac{1}{2}(1 + f_{32}) \\
f_{12} &= \frac{1}{3}(1 + f_{02}) + \frac{1}{3} + \frac{1}{3}(1 + f_{32}) \\
f_{02} &= 1 + f_{12} \\
f_{32} &= \frac{1}{2}(1 + f_{12}) + \frac{1}{2}
\end{align*}
\]

Solve the equations, we have \(f_{22} = 4\).
By symmetric of state 2 and 3, we have $f_{33} = f_{22} = 4$

(c) The relation of $f_{ii}$ and $\pi_i$ is $f_{ii} = \frac{1}{\pi_i}$. This is true for irreducible and aperiodic Markov chain.