1 Solutions for Homework 12

1: \(E(Y|X) = X\theta, \text{Var}(Y|X) = X\theta(1-\theta)\)

\[EY = E(E(Y|X)) = \theta EX = \theta \lambda\]
\[\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) = \theta(1-\theta)\lambda + \theta^2\lambda = \theta \lambda\]

2: \(E(Y|\Theta) = n\Theta, \text{Var}(Y|\Theta) = n\Theta(1-\Theta). \Theta \sim \text{Beta}(a,b), \text{so} \)

\[E(\Theta) = \frac{a}{a+b}, \text{Var}(\Theta) = \frac{ab}{(a+b)^2(a+b+1)}\]

\[E(Y) = E(E(Y|\Theta)) = nE(\Theta) = \frac{na}{a+b}\]

\[\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}[E(Y|X)]\]
\[= n[E\Theta - E(\Theta^2)] + n^2\text{Var}(\Theta)\]
\[= n[E\Theta - (\text{Var}(\Theta) + (E\Theta)^2)] + n^2\text{Var}(\Theta)\]
\[= n\left[\frac{a}{a+b} - \frac{ab}{(a+b)^2(a+b+1)} - \frac{a^2}{(a+b)^2} + \frac{ab}{(a+b)^2(a+b+1)}\right]\]
\[= \frac{nab(a+b+n)}{(a+b)^2(a+b+1)}\]

3: \(E(Y|X) = X, \text{Var}(Y|X) = 1, EY = E[E(Y|X)] = EX = \mu\)

\[\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}[E(Y|X)] = 1 + \text{Var}X = 1 + \sigma^2\]

4: By the weak law of large numbers \(\bar{X} - \rightarrow E(X_1) = 5\), which means for any \(\epsilon > 0\), there exists \(N\), and if \(n > N, P(|\bar{X} - 5| > \epsilon) < 0.001\). Choose \(\epsilon = 1\), then \(P(\bar{X} > 6) \leq P(|\bar{X} - 5| > 1) < 0.001\). So

\[P(X_1 + \cdots + X_n < 6n) = P(\bar{X} < 6) = 1 - P(\bar{X} > 6) > 0.999\]

5: \(EX_n = nP(\text{black is drawn}) = n \ast \frac{1}{n} = 1 \text{ so } \lim_{n \to \infty} E(X_n) = 1\) for any \(\epsilon > 0, P(|X_n| > \epsilon) = P(\text{black is drawn}) = 1/n\rightarrow > 0. \text{ So } X_n \rightarrow > 0 \text{ in probability.}\)

6: (a) \(P(Y_n = 0) = P(X_n = 0) = \frac{\lambda}{\lambda+n}, \text{ So } \lim_{n \to \infty} P(Y_n = 0) = 0\)
(b) \(EY_n = \frac{EX_n}{n} = \frac{1-\lambda/(\lambda+n)}{n\lambda/(\lambda+n)} = 1/\lambda\)
(c) \(\text{Var}(Y_n) = \frac{\text{Var}(X_n)}{n^2} = \frac{1-n\lambda/(\lambda+n)^2}{n^2(\lambda+n)^2} = \frac{\lambda+n}{n\lambda^2}\)
\[ \lim_{n \to \infty} \text{Var}(Y_n) = \frac{1}{\lambda^2} \]

(d) \[ m_{Y_n}(s) = m_{X_n}(s/n) = \frac{\lambda}{\lambda + n(1 - e^{\frac{s}{\lambda}})} \]

By Taylor expansion:
\[ e^{\frac{s}{\lambda}} = 1 + \frac{s}{\lambda} + o\left(\frac{1}{n}\right) \]

So \( \lim_{n \to \infty} n(1 - e^{\frac{s}{\lambda}}) = -s \). Finally
\[ \lim_{n \to \infty} m_{Y_n}(s) = \frac{\lambda}{\lambda - s} \]

\[ m_Y(s) = Ee^{ys} = \int_0^\infty e^{ys} \lambda e^{-\lambda y} dy \]
\[ = \int_0^\infty \lambda e^{-(\lambda-s)y} dy = \frac{\lambda}{\lambda - s} \]

So \( Y_n \to Y \) in distribution.

7: (a) \( \sum_{i=1}^{800} X_i \sim \text{Poisson}(4000) \)

> 1 - ppois(3900, 4000)
[1] 0.9426227

(b) \( E \sum_{i=1}^{800} X_i = 4000, \text{var}(\sum_{i=1}^{800} X_i) = 4000 \)
By the CLT \( \frac{\sum_{i=1}^{800} X_i - 4000}{\sqrt{4000}} \) is close to \( N(0, 1) \)

\[ P\left( \sum_{i=1}^{800} X_i > 3900 \right) = P\left( \frac{\sum_{i=1}^{800} X_i - 4000}{\sqrt{4000}} > \frac{3900 - 4000}{\sqrt{4000}} \right) = 1 - \Phi\left( \frac{-100}{\sqrt{4000}} \right) \]

> 1 - pnorm(-100/sqrt(4000))
[1] 0.9430769