Solution to HW 11

1:  

\[ m(s) = E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} s^2} e^{-(x-s)^2/2+s^2/2} \, dx \]

\[ = e^{s^2/2} \]

2:  

\[ m_X(s) = e^{\mu s} m_Z(s) = e^{\mu s + \frac{s^2}{2}} \]

3:  

\[ m_X(s) = m_{X_1}(s) \cdots m_{X_n}(s) = e^{\mu_1 s + \frac{s^2}{2}} \cdots e^{\mu_n s + \frac{s^2}{2}} \]

\[ = e^{(\sum_{i=1}^{n} \mu_i)s + \frac{(\sum_{i=1}^{n} \sigma_i^2)s^2}{2}} \]

So \( X \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2) \)

4:  \( X \sim Hypergeometric(M = 5, N = 10, n = 4) \) and after four balls added in B, there are 6 blue balls, X white balls, and 4-X red balls in B. So \( Y \mid X \sim Hypergeometric(M = X, N = 10, n = 5) \), so

\[ E[Y \mid X] = nm/N = 5X/10 = X/2 \]

\[ EY = E[E[Y \mid X]] = EX/2 = 0.5 \cdot 4 \cdot 5/10 = 1 \]

5: By symmetric

\[ E(X_1 \mid X = 12) = E(X_2 \mid X = 12) = \cdots = E(X_5 \mid X = 12) \]

\[ E(X_1 + X_2 + \cdots + X_5 \mid X = 12) = E(X \mid X = 12) = 12 \]

So \( E(X_1 \mid X = 12) = E(X_1 + X_2 + \cdots + X_5 \mid X = 12)/5 = 12/5 = 2.4 \)
6:a 

\[ P(X \geq 5) = \sum_{k=5}^{\infty} (1 - \theta)^k \theta = (1 - \theta)^2 = 1/2^5 = 0.0313 \]

By Markov Inequality,

\[ P(X \geq 5) \leq EX/5 = 1/5 \]

b: 

\[ P(|X - 1| \geq 4) = P(X \geq 5) + P(X \leq -3) = P(X \geq 5) = 1/2^5 = 0.313 \]

By Chebychev Inequality,

\[ P(|X - 1| \geq 4) \leq VarX/4^2 = 2/2^4 = 1/8 \]

7: let

\[ X = \begin{cases} 
40 \text{ with prob.} 1/4 \\
0 \text{ with prob.} 3/4 
\end{cases} \]

8: Let

\[ X = \begin{cases} 
16 \text{ with prob.} 1/18 \\
10 \text{ with prob.} 8/9 \\
4 \text{ with prob.} 1/18 
\end{cases} \]