The Poisson($\lambda$) distribution has probability function $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, \ldots$. Random variables $X_1, X_2, \ldots$ are mutually independent and $X_k \sim \text{Poisson}(\frac{1}{2}k)$. Let the sum of the first $n$ of these random variables be $S_n = \sum_{k=1}^{n} X_k$ and let $Y \sim \text{Poisson}(1)$. Note that the moment generating function of the Poisson distribution with parameter $\lambda$ is $m(s) = e^{\lambda(e^s - 1)}$.

1. (5 points) Show that $X_k \xrightarrow{P} 0$ as $k \to \infty$.

Solution: By the definition of convergence in probability, we need to show for any $\epsilon > 0$ that

$$\lim_{k \to \infty} P(|X_k - 0| \geq \epsilon) = 0.$$

It suffices to let $0 < \epsilon < 1$ so that $P(|X_k - 0| \geq \epsilon) = P(X_k \geq 1) = 1 - P(X_k = 0)$. Then,

$$\lim_{k \to \infty} P(|X_k - 0| \geq \epsilon) = \lim_{k \to \infty} 1 - P(X_k = 0)$$

$$= \lim_{k \to \infty} 1 - e^{-(1/2)^k}$$

$$= 1 - e^0$$

$$= 0$$

since $\lim_{k \to \infty} -(1/2)^k = 0$.

2. (5 points) Derive the moment generating function for the Poisson($\lambda$) distribution.

Solution: The moment generating function of random variable $X$ is the expected value of $e^{sX}$. If $X \sim \text{Poisson}(\lambda)$, then

$$m(s) = \mathbb{E}(e^{sX})$$

$$= \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!}$$

$$= e^{-\lambda} e^{e^s \lambda}$$

$$= e^{\lambda(e^s - 1)}.$$

3. (5 points) Find the moment generating function of $S_n$.

Solution: The moment generating function of a sum of independent random variables is the product of the moment generating functions of the individual random variables.

$$m_{S_n}(s) = \prod_{k=1}^{n} e^{(1/2)^k(e^s - 1)} = e^{(1-(1/2)^n)(e^s - 1)}$$

since

$$\sum_{k=1}^{n} (1/2)^k = \frac{(1/2) - (1/2)^{n+1}}{1/2} = 1 - (1/2)^n.$$

Note as well that since $S_n$ is a sum of independent Poisson random variables that $S_n$ is itself a Poisson random variable with mean equal to the sum of the means of the individual $X_k$, and so the moment generating function of $S_n$ has the same form as that for the Poisson distribution.
4. (5 points) Show that $S_n \overset{D}{\to} Y$ as $n \to \infty$.

Solution: We can show convergence in distribution by showing convergence of the moment generating functions.

$$\lim_{n \to \infty} m_{S_n}(s) = \lim_{n \to \infty} e^{(1-(1/2)^n)(e^s-1)} = e^{e^s-1}$$

since

$$\lim_{n \to \infty} 1 - (1/2)^n = 1.$$ 

In addition, the moment generating function of $Y$ is $m_Y(s) = e^{1(e^s-1)}$ since $Y$ is a Poisson(1) random variable. As $\lim_{n \to \infty} m_{S_n}(s) = m_Y(s)$, it follows that $S_n$ converges to $Y$ in distribution.