1. (6 points) A bucket contains ten balls, three of which are red and seven of which are white. Balls are drawn
one at a time at random with each ball being equally likely to be drawn at each draw. When a red ball is
drawn, it is not replaced, but white balls are replaced. Balls are drawn repeatedly until all three red balls
have been selected.

Let $X_i$ be the draw when the $i$th red ball is drawn for $i = 1, 2, 3$. For example, if the sequence of ball colors
is $RWWRWWWR$, then $X_1 = 1$, $X_2 = 4$, and $X_3 = 8$.

Find $E(X_i)$ for $i = 1, 2, 3$. (Hint: You may find it helpful to define other random variables, such as the
number of white balls between red balls.)

Solution: By the description of the problem, there are $W_i$ white balls before the first red one, $W_i$ white balls
after the first red and before the second red, and $W_3$ white balls after the second red ball and before the
third red ball. Since red balls are not replaced, we have $W_1 \sim \text{Geometric}(3/10)$, $W_2 \sim \text{Geometric}(2/9)$, and
$W_3 \sim \text{Geometric}(1/8)$. The mean of the Geometric($\theta$) distribution is $(1 - \theta)/\theta$, so $E(W_1) = 7/3$, $E(W_2) = 7/2$, and
$E(W_3) = 7/1$.

As $X_1 = W_1 + 1$ since $X_1$ counts the first $W_1$ white balls and one red ball, by the linearity of expectation it
follows that $E(X_1) = E(W_1) + 1 = 7/3 + 1 = 10/3$.

Along the same lines, $X_2 = W_1 + W_2 + 2$ since $X_2$ counts all white balls before the second red ball and the first
two red balls. Therefore, $E(X_2) = E(W_1) + E(W_2) + 2 = 7/3 + 7/2 + 2 = 47/6 \approx 7.83$.

Finally, $X_3 = W_1 + W_2 + W_3 + 3$ since $X_3$ counts all white balls before the third red ball and the first three red
balls so that $E(X_3) = E(W_1) + E(W_2) + E(W_3) + 3 = 7/3 + 7/2 + 7 + 3 = 95/6 \approx 15.83$.

2. The random variable $X$ has cumulative distribution function $F(x) = 1 - x^{-3}$ for $x \geq 1$ and $F(x) = 0$ for
$x < 1$.

(a) (3 points) Find the density of $X$.

Solution: Take the derivative of the cumulative distribution function.

$$f(x) = \begin{cases} 3/x^4 & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$$

(b) (4 points) Find $E(X)$.

Solution: Use the definition of expectation for absolutely continuous random variables.

$$\int_1^\infty x(3/x^4) \, dx = \left[ \frac{3}{2x^2} \right]_{x=1}^\infty = 1.5.$$

(c) (3 points) Find $\int_0^\infty P(X > x) \, dx$.

Solution: The easy answer is to recall from the homework that this integral equals $E(X)$, and so by the
previous answer is 1.5. Or, do the integral directly where $P(X > x) = 1$ for $x < 1$ and equals $x^{-3}$ for $x \geq 1$.

$$\int_0^\infty P(X > x) \, dx = \int_0^1 dx + \int_1^\infty x^{-3} \, dx$$

$$= 1 + \left[ \frac{-1}{2x^2} \right]_{x=1}^\infty$$

$$= 1 + (1/2)$$

$$= 1.5$$
(d) (4 points) Find $E((X - 1)^2)$.

Solution: Use the linearity of expectation.

$$E((X - 1)^2) = E(X^2) - 2E(X) + 1$$

We find $E(X^2)$ by direct calculation.

$$E(X^2) = \int_1^\infty (x^2)(3/x^4)dx = -\frac{3}{x}\bigg|_1^\infty = 3$$

Thus,

$$E((X - 1)^2) = 3 - 2(1.5) + 1 = 1.$$