1. Random variables $X$ and $Y$ have a joint probability distribution as follows. $X \sim \text{Poisson}(5)$ so $P(X = k) = e^{-5} 5^k / k!$ for $k = 0, 1, 2, \ldots$. If $X = 0$, then $Y = 0$. Otherwise, roll $X$ fair six-sided dice and let $Y$ be the number of die rolls that are 1’s or 2’s.

For some problems, it will be useful to recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for any $x$.

Solution: Given that $X = x$, $Y$ is the number of successes in $x$ independent Bernoulli trials with success probability $\theta = 1/3$, so $Y \mid X \sim \text{Binomial}(X, 1/3)$ (with $Y = 0$ given that $X = 0$). The joint distribution is then found by applying the multiplication rule.

$$P(X = x, Y = y) = P(X = x) P(Y = y \mid X = x) = e^{-5} \frac{5^x}{x!} \left( \frac{1}{3} \right)^y \left( \frac{2}{3} \right)^{x-y}$$

for $x = 0, 1, 2, \ldots$ and $y = 0, 1, 2, \ldots, x$.

(a) (4 points) What is $P(Y = 0 \mid X = 4)$?

Solution: Since $Y \mid X$ is binomial, this probability is found simply by plugging in to the binomial probability function for $n = 4$ and $\theta = 1/3$.

$$P(Y = 0 \mid X = 4) = \binom{4}{0} \left( \frac{1}{3} \right)^0 \left( \frac{2}{3} \right)^4 = \left( \frac{2}{3} \right)^4 = 0.1975$$

(b) (4 points) What is $P(X = 4 \cap Y = 0)$?

Solution: Plug into the expression for the joint probability distribution given above.

$$P(X = 4 \cap Y = 0) = \frac{e^{-5} 5^4}{4!} \left( \frac{2}{3} \right)^4 = (0.1755)(0.1975) \approx 0.0347$$

(c) (4 points) What is $P(Y = 0)$?

Solution: The random variable $Y$ can be equal to 0 for any possible $X$ since $P(Y = 0 \mid X = k) = (2/3)^k$. Thus, the marginal probability is found by summing the joint density with $Y = 0$ over all possible values of $X$. The unconditional $P(Y = 0)$ is the weighted average of the conditional probabilities $P(Y = 0 \mid X = k)$ weighted by the probabilities $P(X = k)$ — the law of total probability in action to find a marginal distribution from a joint probability distribution.

$$P(Y = 0) = \sum_{k=0}^{\infty} P(X = k; Y = 0)$$

$$= \sum_{k=0}^{\infty} e^{-5} \frac{5^k}{k!} \left( \frac{2}{3} \right)^k$$

$$= e^{-5} \sum_{k=0}^{\infty} \frac{(10/3)^k}{k!}$$

$$= e^{-5} e^{10/3}$$

$$= e^{-5/3}$$

$$= 0.1889$$
(d) (3 points) Are \( X \) and \( Y \) independent? Briefly explain why or why not.

Solution: \( X \) and \( Y \) are not independent since it is not true that \( P(X = x, Y = y) = P(X = x)P(Y = y) \) for all \( x \) and \( y \). In particular, \( P(X = 4, Y = 0) \neq P(X = 4)P(Y = 0) \). Or, even more simply, \( P(X = 0, Y = 1) = 0 < P(X = 0)P(Y = 1) \).

(e) (2 points) Find the marginal distribution of \( Y \).

Solution: To find the marginal distribution of \( Y \), sum the joint distribution over the possible values of \( X \). Note that \( Y = y \) implies that \( X \geq y \) since you cannot get \( y \) successes from fewer than \( y \) Bernoulli trials. Also, it is necessary to factor a constant out of the sum so that the remaining sum is in the form of an exponential series.

\[
P(Y = y) = \sum_{x=y}^{\infty} \frac{e^{-5/3}x}{x!} \left( \frac{1}{3} \right)^y \left( \frac{2}{3} \right)^{x-y} = \sum_{x=y}^{\infty} \frac{e^{-5/3}x}{x!} \frac{x!}{y!(x-y)!} \left( \frac{1}{3} \right)^y \left( \frac{2}{3} \right)^{x-y} = \frac{e^{-5/3}y}{y!} \sum_{x=y}^{\infty} (10/3)^{x-y} = \frac{e^{-5/3}y}{y!} e^{10/3} = \frac{e^{-5/3}(5/3)^y}{y!}
\]

Notice that \( Y \sim \text{Poisson}(5/3) \).

The more general fact is if \( X \sim \text{Poisson}(\lambda) \) and \( Y \mid X \sim \text{Binomial}(X, \theta) \), then \( Y \sim \text{Poisson}(\lambda\theta) \).

2. (3 points) Select a point uniformly at random from the unit disk,

\[
D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}
\]

and let \( X \) and \( Y \) be the \( x \)- and \( y \)-coordinates of this point. In other words, \( X \) and \( Y \) have joint density \( f_{X,Y}(x, y) = 1/\pi \) if \( x^2 + y^2 < 1 \) and 0 otherwise. Let \( W = X/Y \). Find the density of \( W \).

Solution: To find the density of \( W \), we need to find the joint density of \( W \) and another random variable, say \( Z = X \), and then integrate out \( Z \). This is a transformation \( (W, Z) = h(X, Y) \).

The inverse transformation between these random variables is \( (X, Y) = h^{-1}(W, Z) \) which is solved by \( X = Z \) and \( Y = Z/W \). Notice that the possible values of \( W \) range from \(-\infty \) to \( \infty \) and the possible values of \( Z \) range from \(-1 \) to \( 1 \), but the pair \((W, Z)\) is constrained by

\[
Z^2 + (Z/W)^2 < 1 \quad \text{or} \quad \frac{Z^2(1 + W^2)}{W^2} < 1.
\]

The joint density of \( W \) and \( Z \) is then given by

\[
f_{W,Z}(w, z) = \frac{1}{|J(h^{-1}(w, z))|} f_{X,Y}(h^{-1}(w, z)).
\]

where \( J \) is the Jacobian derivative of the transformation. The Jacobian derivative is

\[
J = \begin{vmatrix} \frac{\partial(W, Z)}{\partial(X, Y)} \end{vmatrix} = \begin{vmatrix} 1/Y & 1 \\ -X/Y^2 & 0 \end{vmatrix} = \frac{X}{Y^2} = \frac{Z}{(Z/W)^2} = \frac{W^2}{Z}.
\]
Therefore, the joint density of $W$ and $Z$ is

$$f_{W,Z}(w, z) = \frac{1}{|w^2/z|} \times \frac{1}{\pi} = \frac{z}{\pi w^2}$$

for $z^2(1 + w^2)/w^2 < 1$. Notice that for fixed $w$, $-\sqrt{w^2/(1 + w^2)} < z < \sqrt{w^2/(1 + w^2)}$.

The density of $W$ is found by integrating out $Z$ from the joint density.

$$f_W(w) = \int_{-\sqrt{w^2/(1 + w^2)}}^{\sqrt{w^2/(1 + w^2)}} \frac{z}{\pi w^2} \, dz$$

$$= \frac{1}{\pi w^2} \left( \frac{z^2}{2} \bigg|_{z=-\sqrt{w^2/(1 + w^2)}}^{\sqrt{w^2/(1 + w^2)}} \right)$$

$$= \frac{1}{\pi w^2} \frac{w^2}{1 + w^2}$$

for $-\infty < w < \infty$. So, $W$ has the Cauchy distribution.

Furthermore, it is possible to prove the more general fact that if $X$ and $Y$ have any joint distribution that is circularly symmetric, such as the bivariate distribution of two independent normal random variables, then $X/Y$ has the Cauchy distribution.

Of course, other choices for $Z$ are possible. The choice $Z = Y$ implies that $X = WZ$ and following this up with the method above leads to a slightly easier solution.

The choice $Z = XY$ is not best as the points $(X, Y)$ and $(-X, -Y)$ will have the same ratio and product, and so the inverse function will not be one-to-one, and this would need to be corrected for using methods beyond those discussed in lecture.